Lecture 8: Harmonic Loads

Reading materials: Sections 3.1, 3.2 and 3.3

1. Introduction

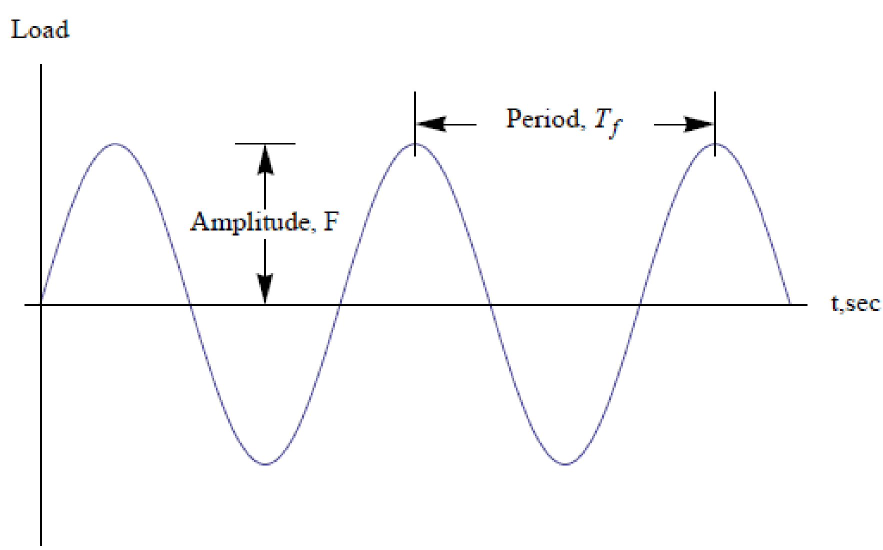
When an applied load varies as a sine or a cosine function, it is called harmonic loading.

Sine loading:

$$f(t) = F \sin \omega_f t$$

Cosine loading:

$$f(t) = F \cos \omega_f t$$



The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature.

The response of a system to a harmonic excitation (loading) is called *harmonic response*.

- The nonperiodic excitation excitation may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response*.

- If the frequency of excitation coincides with the natural frequency of the system, the response of the system will be very large. This condition is known as *resonance*, which should be avoided to prevent failure of the system.

- The examples of harmonic motion include the vibration produced by an unbalanced rotating machine, the oscillations of a tall chimney due to vortex shedding in a steady wind, and the vertical motion of an automobile on a sinusoidal road surface.

2. Equations of motion

- General equations of motion:

$$m\ddot{u} + c\dot{u} + ku = f(t); \quad u(0) = u_0; \quad \dot{u}(0) = v_0$$

- Loading expressed as complex exponential function

Equations of motion

$$m\ddot{u} + c\dot{u} + ku = F e^{i\omega_f t}; \quad u(0) = u_0; \quad \dot{u}(0) = v_0$$

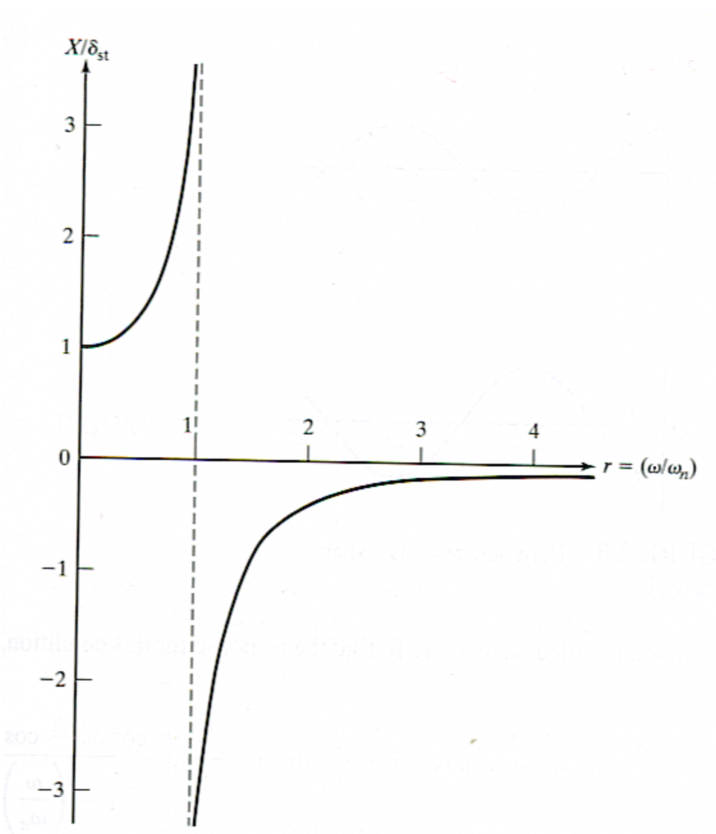
- Multi degree of freedom systems subjected to harmonic loading

Equations of motion

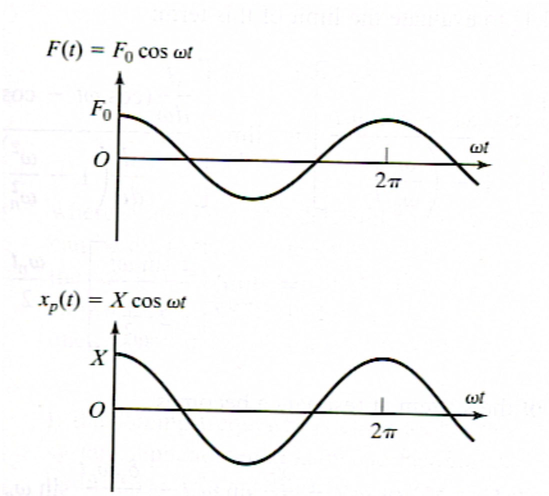
$$\boldsymbol{m}\ddot{\boldsymbol{u}} + \boldsymbol{c}\dot{\boldsymbol{u}} + \boldsymbol{k}\boldsymbol{u} = \boldsymbol{F} e^{i\omega_f t}; \quad \boldsymbol{u}(0) = \boldsymbol{u}_0; \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$$

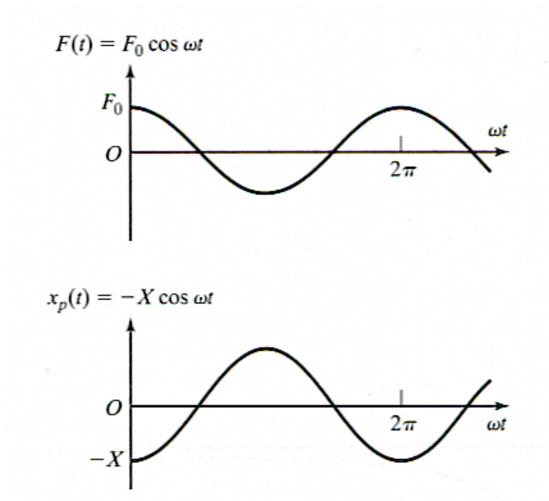
- Analytical solutions will be obtained for single DOF systems subjected to harmonic loading. The solution of multiple DOF systems will be obtained via modal superposition approach.

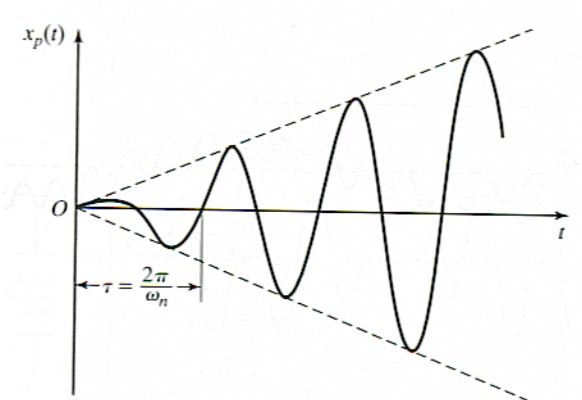
3. Response of an undamped system under harmonic force



(Note: ω here is the load frequency ω_f in our text book, while ω_n is the system natural frequency ω in our text book.)

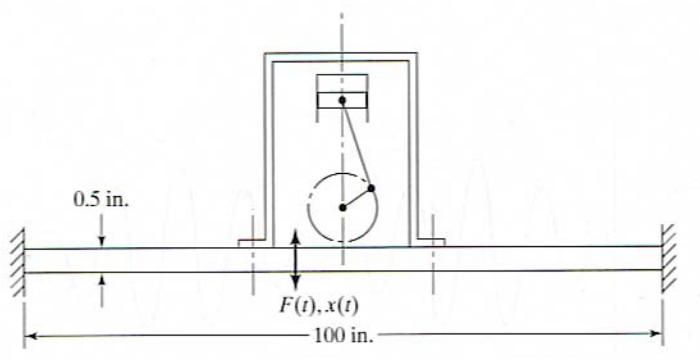






(Note: ω here is the load frequency ω_f in our text book, while ω_n is the system natural frequency ω in our text book.)

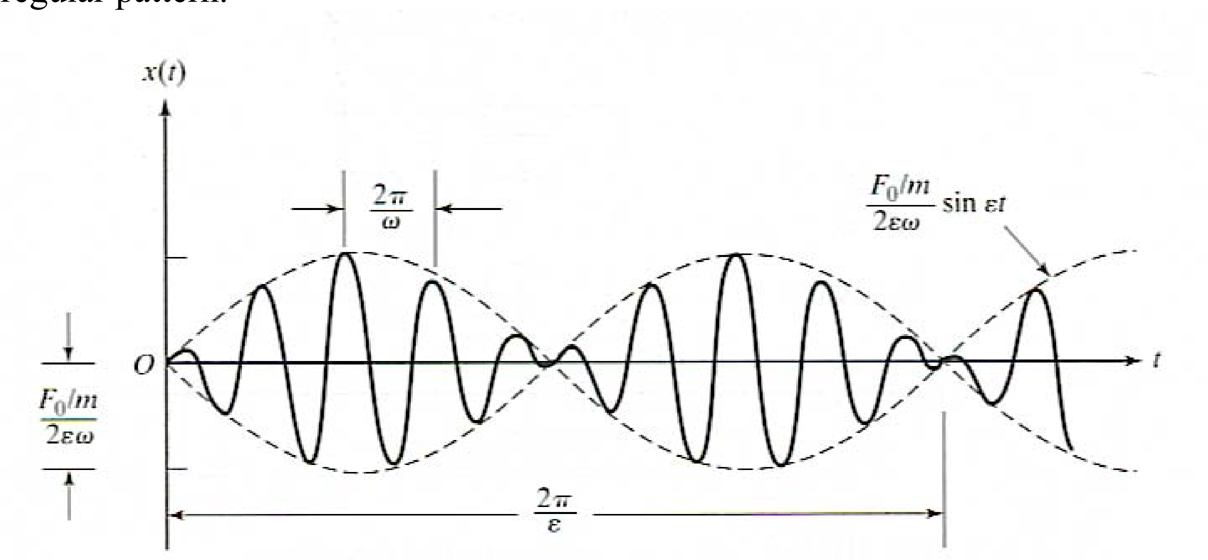
Example: A reciprocating pump, weighing 150lb, is mounted at the middle of a steel plate of thickness 0.5 in., width 20in., and length 100 in., clamped along two edges. During operation of the pump, the plate is subjected to a harmonic force, $F(t)=50\cos 62.832t$ lb. Find the amplitude of vibration of the plate



4. Beating phenomenon (Optional)

- If the load frequency, ω_f, is close to, not exactly equal to, the natural frequency of the system, ω, a phenomenon known as beating may occur.

- In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern.



(Note: ω here is the load frequency ω_f in our text book, while ω_n is the system natural frequency ω in our text book.)

It can be observed that the *sin* $\omega_f t$ curve will go through several cycles, while the *sin* εt wave goes through a single cycle. Therefore, the amplitude builds up and dies down continuously.

5. Response of a damped system under harmonic force in exponential form (Optional)

Loading expressed as complex exponential function

Equations of motion

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = F\,e^{i\,\omega_f\,t}; \quad u(0) = u_0 \quad \dot{u}(0) = v_0$$

Superposition the complete solution is the sum of the solution to free vibration due to initial conditions and the particular solution due to load, generally.

$$u(t) = u_t(t) + u_s(t)$$

For the free vibration solution

$$m\,\ddot{u}_t(t) + c\,\dot{u}_t(t) + k\,u_t(t) = 0; \quad u(0) = u_0 \quad \dot{u}(0) = v_0$$

$$u_t(t) = e^{-\xi\,\omega\,t}(A\cos\omega_d\,t + B\sin\omega_d\,t)$$

$$\omega_d = \omega\sqrt{1-\xi^2}\,; \quad \omega = \sqrt{k/m}\,; \quad \xi = \frac{c}{2\,m\,\omega}$$

For the particular solution:

$$m\,\ddot{u}_s(t) + c\,\dot{u}_s(t) + k\,u_s(t) = F\,e^{i\,\omega_f\,t}$$

Assume

$$u_s(t) = U\,e^{i\,\omega_f\,t}$$

Then

$$-\omega_f^2 m\, U\, e^{i\,\omega_f t} + i\,\omega_f\, c\, U\, e^{i\,\omega_f t} + k\, U\, e^{i\,\omega_f t} = F\, e^{i\,\omega_f t}$$

$$U = \frac{F}{-\omega_f^2 m + i\,\omega_f\, c + k} = \frac{F/k}{-\omega_f^2 (m/k) + i\,\omega_f (c/k) + 1}$$

Since

$$\frac{m}{k} = \frac{1}{\omega^2} \text{ and } \frac{c}{k} = \frac{2\, m\, \omega\, \xi}{m\, \omega^2} = \frac{2\,\xi}{\omega}$$

We have

$$U = \frac{F/k}{1 - (\omega_f/\omega)^2 + i\, 2\,\xi(\omega_f/\omega)}$$

Let

$$r = \omega_f \big/ \omega$$

Then,

$$U = \frac{F/k}{1 - r^2 + i\,2\,\xi\, r} = \frac{F/k}{1 - r^2 + i\,2\,\xi\, r}\,\frac{1 - r^2 - i\,2\,\xi\, r}{1 - r^2 - i\,2\,\xi\, r} = \frac{F/k}{(1 - r^2)^2 + (2\,\xi\, r)^2}\left(1 - r^2 - i\,2\,\xi\, r\right)$$

Finally

$$u_s(t) = \frac{F/k}{(1 - r^2)^2 + (2\,\xi\, r)^2}\left(1 - r^2 - i\,2\,\xi\, r\right)\, e^{i\,\omega_f t}$$

Solution for sine and cosine loading

$$u_s(t) = \frac{F/k}{(1 - r^2)^2 + (2\,\xi\, r)^2}\left(1 - r^2 - i\,2\,\xi\, r\right)\left(\cos\,\omega_f t + i\sin\,\omega_f t\right)$$

Separating the real and the complex parts

$$u_s(t) = \frac{F/k}{(1-r^2)^2 + (2\xi r)^2}\left[(1 - r^2)\cos\omega_f t + 2\xi r \sin\omega_f t + i\left((1 - r^2)\sin\omega_f t - 2\xi r\cos\omega_f t\right)\right]$$

Steady-state solution for cosine loading

$$u_s(t) = \frac{F/k}{(1-r^2)^2 + (2\xi r)^2}\left[(1 - r^2)\cos\omega_f t + 2\xi r \sin\omega_f t\right]$$

Complete solution for cosine loading

$$u(t) = u_t(t) + u_s(t)$$

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F/k}{\gamma^2}\left[(1 - r^2)\cos\omega_f t + 2\xi r \sin\omega_f t\right]$$

$$\omega_d = \omega\sqrt{1-\xi^2}\,; \quad \omega = \sqrt{k/m}\,; \quad \xi = \frac{c}{2m\omega}; \quad r = \frac{\omega_f}{\omega}$$

$$\theta = \tan^{-1}\left(\frac{2r\xi}{1-r^2}\right); \quad \gamma = \sqrt{(1 - r^2)^2 + (2\xi r)^2}$$

Steady-state solution for sine loading

$$u_s(t) = \frac{F/k}{(1-r^2)^2 + (2\xi r)^2}\left[(1 - r^2)\sin\omega_f t - 2\xi r \cos\omega_f t\right]$$

Complete solution for sine loading

$$u(t) = e^{-\xi\omega t}(A\cos\omega_d t + B\sin\omega_d t) + \frac{F/k}{(1-r^2)^2 + (2\xi r)^2}\left[(1 - r^2)\sin\omega_f t - 2\xi r \cos\omega_f t\right]$$

6. Response of a damped system under a cosine load

- Any amount of damping ($\xi > 0$) reduces the magnification factor for all values of the forcing frequency.

- The amplitude of forced vibration becomes smaller with increasing values of forcing frequency

- For an undamped system ($\xi = 0$), the excitation and response are in phase (the phase angle is 0) for $0 < r < 1$ and out of phase (the phase angle is 180^o) for $r > 1$.

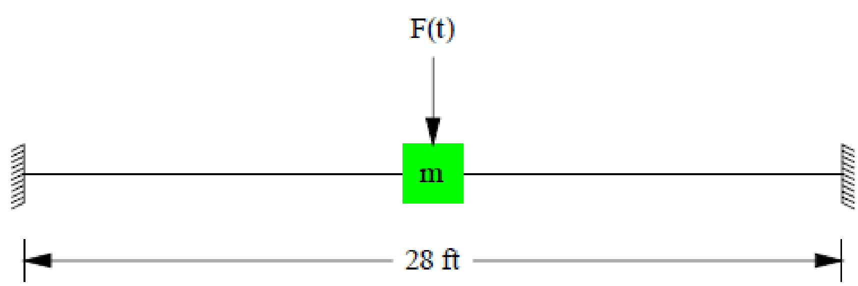
- For $\xi > 0$ and $r > 1$, the phase angle is given by $90^o < \phi < 180^o$, implying that the response leads the excitation.

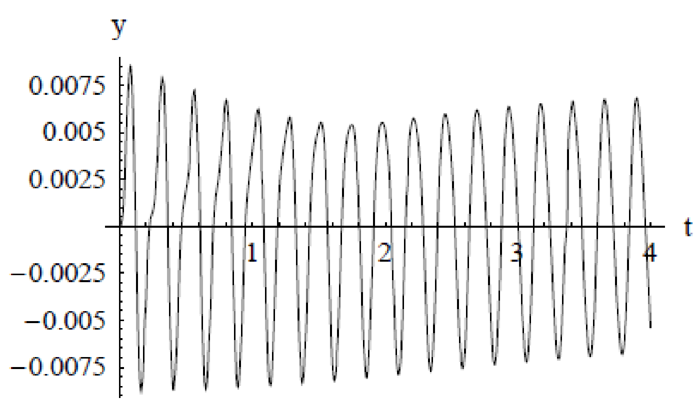
- For $\xi > 0$ and $0 < r < 1$, the phase angle is given by $0^o < \phi < 90^o$, implying that the response lags the excitation.

- For $\xi > 0$ and $r = 1$, the phase angle is given by $\phi = 90^o$, implying that the phase difference between the excitation and the response is 90^o

- For $\xi > 0$ and large values of r, the phase angle approach 180^o, implying that the response and the excitation are out of phase.

7. Example

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