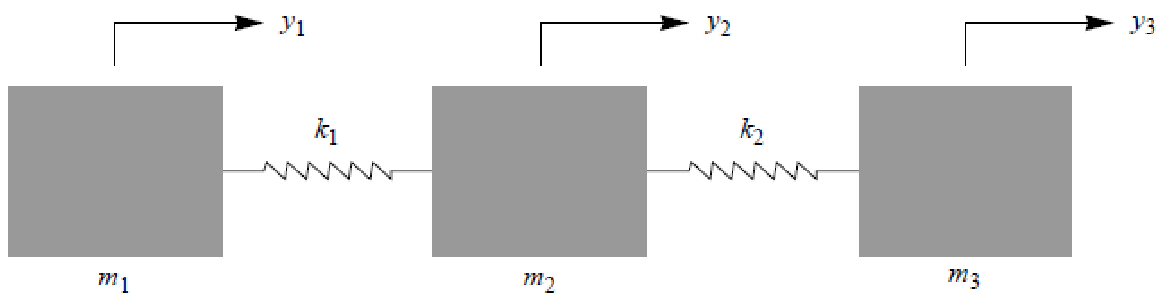
Lecture 7: Systems involving zero or repeat Frequencies

Reading materials: Sections 2.4 and 2.5

1. Systems involving zero frequency

- Some possible mode shapes may not involve any deformation. They are called rigid body modes. The corresponding frequencies are zero.

- Example: an unrestrained three spring-mass system

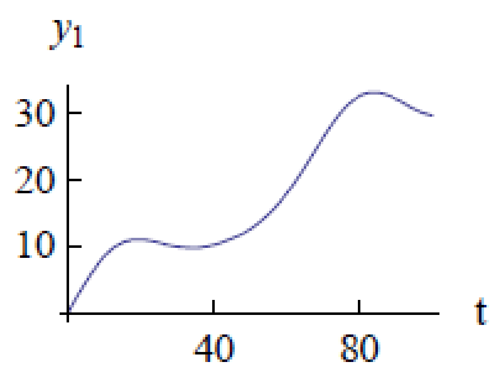


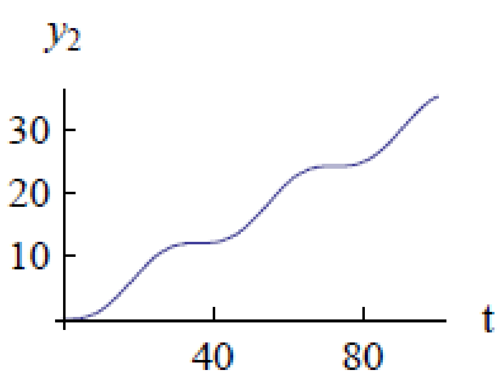
Exact solution

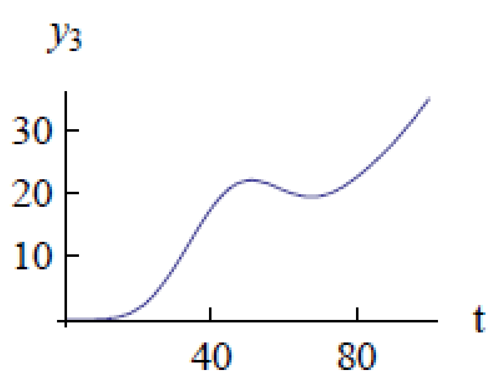
$$y_1(t) = \frac{t}{3} + 5\sin\left(\frac{t}{10}\right) + \frac{5\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3\sqrt{3}}$$

$$y_2(t) = \frac{t}{3} - \frac{10\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3\sqrt{3}}$$

$$y_3(t) = \frac{t}{3} - 5\sin\left(\frac{t}{10}\right) + \frac{5\sin\left(\frac{\sqrt{3}\,t}{10}\right)}{3\sqrt{3}}$$

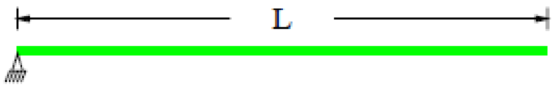






Modal superposition solution

Some dynamic systems exhibit rigid-body modes that are characterized by zero natural frequencies.



The beam is not properly restrained. Its first mode is a rigid body mode in which the beam pivots around its left support.

Generally, the uncoupled modal equations are

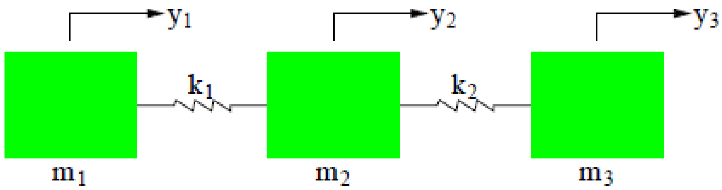
$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = \frac{1}{M_i} F_i(t) \quad i = 1, 2, \ldots$$

Since the frequency is zero, the corresponding uncoupled modal equation is

$$\ddot{z}_i(t) = \frac{1}{M_i} F_i(t) \qquad z_i(0) = z_{i0}; \quad \dot{z}_i(0) = \dot{z}_{i0}$$

For free vibration:

Example



$m_1 = 50; \quad m_2 = 100; \quad m_3 = 150; \quad k_1 = 1000; \quad k_2 = 500;$

$$\boldsymbol{m} = \begin{pmatrix} 50 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 150 \end{pmatrix}; \quad \boldsymbol{k} = \begin{pmatrix} 1000 & -1000 & 0 \\ -1000 & 1500 & -500 \\ 0 & -500 & 500 \end{pmatrix}; \quad \boldsymbol{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\boldsymbol{u}^0 = \{0, 0, 0\}$ $\qquad \boldsymbol{v}^0 = \{1, 0, 0\}$

	Eigenvalue	Frequency (rad/s)	Mode shape		
1	-9.92523×10^{-17}	0.	0.057735	0.057735	0.057735
2	6.22985	2.49597	−0.0722489	−0.0497439	0.0572456
3	32.1035	5.66599	−0.10699	0.0647473	−0.00750167

$M_i = \boldsymbol{\phi}_i^T \boldsymbol{m} \boldsymbol{\phi}_i$: $\quad \{1, 1, 1\}$

$K_i = \boldsymbol{\phi}_i^T \boldsymbol{k} \boldsymbol{\phi}_i$: $\quad \{0., 6.22985, 32.1035\}$

$F_i = \boldsymbol{\phi}_i^T \boldsymbol{f}$: $\quad \{0., 0., 0.\}$

$\ddot{z}_1 + 0 = 0; \quad z_1(0) = 0.; \quad \dot{z}_1(0) = 2.88675$

$\ddot{z}_2 + 6.22985\, z_2 = 0; \quad z_2(0) = 0.; \quad \dot{z}_2(0) = -3.61245$

$\ddot{z}_3 + 32.1035\, z_3 = 0; \quad z_3(0) = 0.; \quad \dot{z}_3(0) = -5.34948$

$$z_1(t) = 2.88675\,t$$

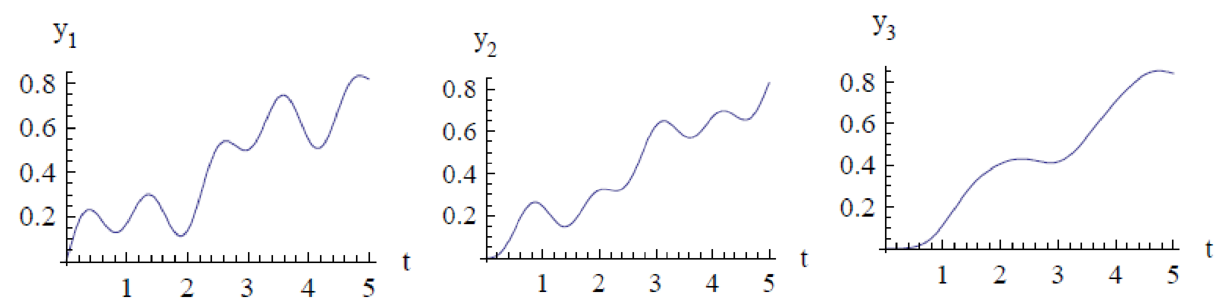
$$z_2(t) = -1.44731\sin(2.49597\,t)$$

$$z_3(t) = -0.944137\sin(5.66599\,t)$$

$$y_1(t) = 0.166667\,t + 0.104567\sin(2.49597\,t) + 0.101013\sin(5.66599\,t)$$

$$y_2(t) = 0.166667\,t + 0.071995\sin(2.49597\,t) - 0.0611303\sin(5.66599\,t)$$

$$y_3(t) = 0.166667\,t - 0.0828523\sin(2.49597\,t) + 0.00708261\sin(5.66599\,t)$$



2. Systems involving repeated frequency

- In a special case, all frequencies of a dynamic system are not unique.
- Example

$$\boldsymbol{m} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad \boldsymbol{k} = \begin{pmatrix} 44 & -24 & 0 \\ -24 & 24 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$y_1(t) = \frac{8}{13}\sqrt{6}\,\sin\left(\sqrt{\frac{2}{3}}\,t\right) - \frac{3\sin\left(\sqrt{5}\,t\right)}{13\sqrt{5}}$$

$$y_2(t) = \frac{12}{13}\sqrt{6}\,\sin\left(\sqrt{\frac{2}{3}}\,t\right) + \frac{2\sin\left(\sqrt{5}\,t\right)}{13\sqrt{5}}$$

$$y_3(t) = \frac{3\sin\left(\sqrt{5}\,t\right)}{\sqrt{5}}$$