Lecture 6: Modal Superposition

Reading materials: Section 2.3

1. Introduction

Exact solution of the free vibration problems is

$$y(t) = \sum_m (A_i \cos \omega_i t + B_i \sin \omega_i t)\, \phi^{(i)}$$

where coefficients can be determined from the initial conditions.

The method is not practical for large systems since two unknown coefficients must be introduced for each mode shape.

Modal superposition is a powerful idea of obtaining solutions. It is applicable to both free vibration and forced vibration problems.

The basic idea

To use free vibrations mode shapes to uncouple equations of motion.
The uncoupled equations are in terms of new variables called the modal coordinates.
Solution for the modal coordinates can be obtained by solving each equation independently.
A superposition of modal coordinates then gives solution of the original equations.

Notices
It is not necessary to use all mode shapes for most practical problems.
Good approximate solutions can be obtained via superposition with only first few mode shapes.

2. Orthogonality of undamped free vibration mode shapes

An n degree of freedom system has n natural frequencies and n corresponding mode shapes.

$$[\boldsymbol{k} - \lambda_i \boldsymbol{m}]\,\boldsymbol{\phi}^{(i)} = \mathbf{0}$$

$$\boldsymbol{k}\,\boldsymbol{\phi}_i = \lambda_i\,\boldsymbol{m}\,\boldsymbol{\phi}_i; \quad i = 1, 2, \ldots, n$$

Mass orthogonality:

$$\boldsymbol{\phi}_j^T\,\boldsymbol{m}\,\boldsymbol{\phi}_i = 0; \quad i \neq j$$

Proof:

Mass nomalization:

$$\boldsymbol{\phi}_i = \frac{1}{\sqrt{\hat{\boldsymbol{\phi}}_i^T\,\boldsymbol{m}\,\hat{\boldsymbol{\phi}}_i}}\,\hat{\boldsymbol{\phi}}_i$$

Stiffness orthogonality:

$$\boldsymbol{\phi}_j^T\, \boldsymbol{k}\, \boldsymbol{\phi}_i = 0; \quad i \neq j$$

Proof:

3. Modal superposition for undamped systems – Uncoupling of the Equations of motion

Equations of motion of an undamped multi-degree of freedom system

$$\boldsymbol{m}\, \ddot{\boldsymbol{y}}(t) + \boldsymbol{k}\, \boldsymbol{y}(t) = \boldsymbol{f}\,(t); \quad \boldsymbol{y}(0) = \boldsymbol{u}^0 \text{ and } \dot{\boldsymbol{y}}(0) = \boldsymbol{v}^0$$

The displacement vector can be written as a linear combination of the mode shape vectors.

$$\boldsymbol{y}\,(t) = z_1\,(t)\,\boldsymbol{\phi}_1 + z_2(t)\,\boldsymbol{\phi}_2 + \ldots + z_n\,(t)\,\boldsymbol{\phi}_n$$

or in matrix form,

$$\boldsymbol{y}\,(t) = \boldsymbol{\Phi}\, \boldsymbol{z}\,(t)$$

$$\boldsymbol{\Phi} = (\, \boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \ldots \quad \boldsymbol{\phi}_n \,)$$

$$\boldsymbol{z} = (\, z_1 \quad z_2 \quad \ldots \quad z_n \,)^T$$

Then, the equations of motion

$$\boldsymbol{m}\,\Phi\,\ddot{z}(t) + \boldsymbol{k}\,\Phi\,z(t) = \boldsymbol{f}\,(t)$$

$$\Phi^{T}\,\boldsymbol{m}\,\Phi\,\ddot{z}(t) + \Phi^{T}\,\boldsymbol{k}\,\Phi\,z(t) = \Phi^{T}\,\boldsymbol{f}\,(t)$$

First term becomes a modal mass matrix using mass orthogonalitys

$$\Phi^{T}\,\boldsymbol{m}\,\Phi = \begin{pmatrix} \phi_1^T \\ \phi_2^T \\ \vdots \\ \phi_n^T \end{pmatrix} \boldsymbol{m}(\,\phi_1 \quad \phi_2 \quad \cdots \quad \phi_n\,) = \begin{pmatrix} \phi_1^T\,\boldsymbol{m}\,\phi_1 & \phi_1^T\,\boldsymbol{m}\,\phi_2 & \cdots & \phi_1^T\,\boldsymbol{m}\,\phi_n \\ \phi_2^T\,\boldsymbol{m}\,\phi_1 & \phi_2^T\,\boldsymbol{m}\,\phi_2 & \cdots & \phi_2^T\,\boldsymbol{m}\,\phi_n \\ \vdots & \vdots & \cdots & \vdots \\ \phi_n^T\,\boldsymbol{m}\,\phi_1 & \phi_n^T\,\boldsymbol{m}\,\phi_2 & \cdots & \phi_n^T\,\boldsymbol{m}\,\phi_n \end{pmatrix}$$

$$\Phi^{T}\,\boldsymbol{m}\,\Phi = \begin{pmatrix} \phi_1^T\,\boldsymbol{m}\,\phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2^T\,\boldsymbol{m}\,\phi_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \phi_n^T\,\boldsymbol{m}\,\phi_n \end{pmatrix} \equiv \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & M_n \end{pmatrix}$$

Second term becomes a stiffness matrix using stiffness orthogonality

$$\Phi^{T}\,\boldsymbol{k}\,\Phi = \begin{pmatrix} \phi_1^T\,\boldsymbol{k}\,\phi_1 & 0 & \cdots & 0 \\ 0 & \phi_2^T\,\boldsymbol{k}\,\phi_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \phi_n^T\,\boldsymbol{k}\,\phi_n \end{pmatrix} \equiv \begin{pmatrix} K_1 & 0 & \cdots & 0 \\ 0 & K_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & K_n \end{pmatrix}$$

Here is the modal load vector

$$\Phi^{T}\,\boldsymbol{f}\,(t) \equiv \boldsymbol{F}(t) \quad \Longrightarrow \quad F_i = \phi_i^T\,\boldsymbol{f};\; i = 1,\,\ldots,\,n$$

The equations of motion are uncoupled and known as the modal equations

$$\begin{pmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & M_n \end{pmatrix}\begin{pmatrix} \ddot{z}_1(t) \\ \ddot{z}_2(t) \\ \vdots \\ \ddot{z}_n(t) \end{pmatrix} + \begin{pmatrix} K_1 & 0 & \dots & 0 \\ 0 & K_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & K_n \end{pmatrix}\begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

or

$$M_i\, \ddot{z}_i(t) + K_i\, z_i(t) = F_i(t); \quad i = 1, 2, \dots$$

$$M_i = \boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{\phi}_i; \quad K_i = \boldsymbol{\phi}_i^T\, \boldsymbol{k}\, \boldsymbol{\phi}_i; \quad F_i = \boldsymbol{\phi}_i^T\, \boldsymbol{f}$$

Recall natural frequencies

$$\boldsymbol{k}\, \boldsymbol{\phi}_i = \lambda_i\, \boldsymbol{m}\, \boldsymbol{\phi}_i \quad \Longrightarrow \quad \boldsymbol{\phi}_i^T\, \boldsymbol{k}\, \boldsymbol{\phi}_i = \lambda_i\, \boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{\phi}_i$$

$$\Longrightarrow \quad K_i = \lambda_i\, M_i \quad \Longrightarrow \quad \lambda_i \equiv \omega_i^2 = \frac{K_i}{M_i}$$

Then

$$\ddot{z}_i(t) + \omega_i^2\, z_i(t) = \frac{1}{M_i}\, F_i(t) \quad i = 1, 2, \dots$$

Obviously, each modal equation represents an equivalent single degree of freedom system.

Rewrite the initial conditions for the modal equations

$$z_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{u}^0\right)$$

$$\dot{z}_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{v}^0\right)$$

Finally, the modal equations are

$$\ddot{z}_i(t) + \omega_i^2\, z_i(t) = \frac{1}{M_i}\, F_i(t) \quad i = 1, 2, \ldots$$

$$z_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{u}^0\right)$$

$$\dot{z}_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{v}^0\right)$$

$$M_i = \boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{\phi}_i; \quad K_i = \boldsymbol{\phi}_i^T\, \boldsymbol{k}\, \boldsymbol{\phi}_i; \quad F_i = \boldsymbol{\phi}_i^T\, \boldsymbol{f}\,; \quad \omega_i = \sqrt{K_i / M_i}$$

4. Modal superposition for undamped systems – Solution of the modal equations

For free vibrations, the modal equations are:

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = 0$$

For each equation, the solution is

$$z_i(t) = z_i(0)\cos\omega_i t + \frac{\dot{z}_i(0)}{\omega_i}\sin\omega_i t$$

or

$$z_i(t) = Z_i \sin(\omega_i t + \theta_i) = Z_i \cos(\omega_i t - \psi_i)$$

where

$$Z_i = \sqrt{[z_i(0)]^2 + \left(\frac{\dot{z}_i(0)}{\omega_i}\right)^2}$$

$$\theta_i = \tan^{-1}\left(z_i(0) \Big/ \left(\frac{\dot{z}_i(0)}{\omega_i}\right)\right); \quad \psi_i = \tan^{-1}\left(\left(\frac{\dot{z}_i(0)}{\omega_i}\right) \Big/ z_i(0)\right)$$

Then, the solution for the original equations of motion is

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}\,\boldsymbol{z}(t) \equiv z_1(t)\,\boldsymbol{\phi}_1 + z_2(t)\,\boldsymbol{\phi}_2 + \ldots + z_n(t)\,\boldsymbol{\phi}_n$$

Indeed, the above solution is the exact solution. The approximate solution can be obtained via using the first few mode shapes.

$$\boldsymbol{y}(t) \approx z_1(t)\,\boldsymbol{\phi}_1 + z_2(t)\,\boldsymbol{\phi}_2 + \ldots + z_m(t)\,\boldsymbol{\phi}_m; \quad m << \mathrm{n}$$

The above equations are general expressions for both free vibration and forced vibration.

For forced vibration, $z_i(t)$ could be obtained from the solution of one DOF forced vibration.

5. Examples

$$\boldsymbol{m} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}; \qquad \boldsymbol{k} = \begin{pmatrix} 10 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 5 \end{pmatrix}; \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{u}^0 = \{0.025, 0.02, 0.01, 0.001\}$$

$$\boldsymbol{v}^0 = \{0, 0, 0, 0\}$$

Eigenvalues, frequencies, and mode shapes

	Eigenvalue	Frequency (rad/s)	Mode shape			
1	0.150768	0.388289	0.114007	0.214263	0.288675	0.328269
2	1.25	1.11803	0.288675	0.288675	0.	−0.288675
3	2.93412	1.71293	0.328269	−0.114007	−0.288675	0.214263
4	4.41511	2.10122	−0.214263	0.328269	−0.288675	0.114007

a. Uncoupling equations of motion

$$\ddot{z}_i(t) + \omega_i^2 z_i(t) = \frac{1}{M_i} F_i(t) \quad i = 1, 2, \ldots$$

$\mathrm{M_i} = \boldsymbol{\phi}_i^T \boldsymbol{m} \, \boldsymbol{\phi}_i$: $\{1, 1, 1, 1\}$

$\mathrm{K_i} = \boldsymbol{\phi}_i^T \boldsymbol{k} \, \boldsymbol{\phi}_i$: $\{0.150768, 1.25, 2.93412, 4.41511\}$

$\mathrm{F_i} = \boldsymbol{\phi}_i^T \boldsymbol{f}$: $\{0., 0., 0., 0.\}$

I.C.s:

$(\boldsymbol{\phi}_i^T \boldsymbol{m} \, \boldsymbol{u}^0)/\mathrm{M_i}$: $\{0.0414018, 0.0508068, 0.0130164, -0.00625569\}$

$(\boldsymbol{\phi}_i^T \boldsymbol{m} \, \boldsymbol{v}^0)/\mathrm{M_i}$: $\{0., 0., 0., 0.\}$

Modal equations:

$\ddot{z}_1 + 0.150768\, z_1 = 0;$ $z_1(0) = 0.0414018;$ $\dot{z}_1(0) = 0.$

$\ddot{z}_2 + 1.25\, z_2 = 0;$ $z_2(0) = 0.0508068;$ $\dot{z}_2(0) = 0.$

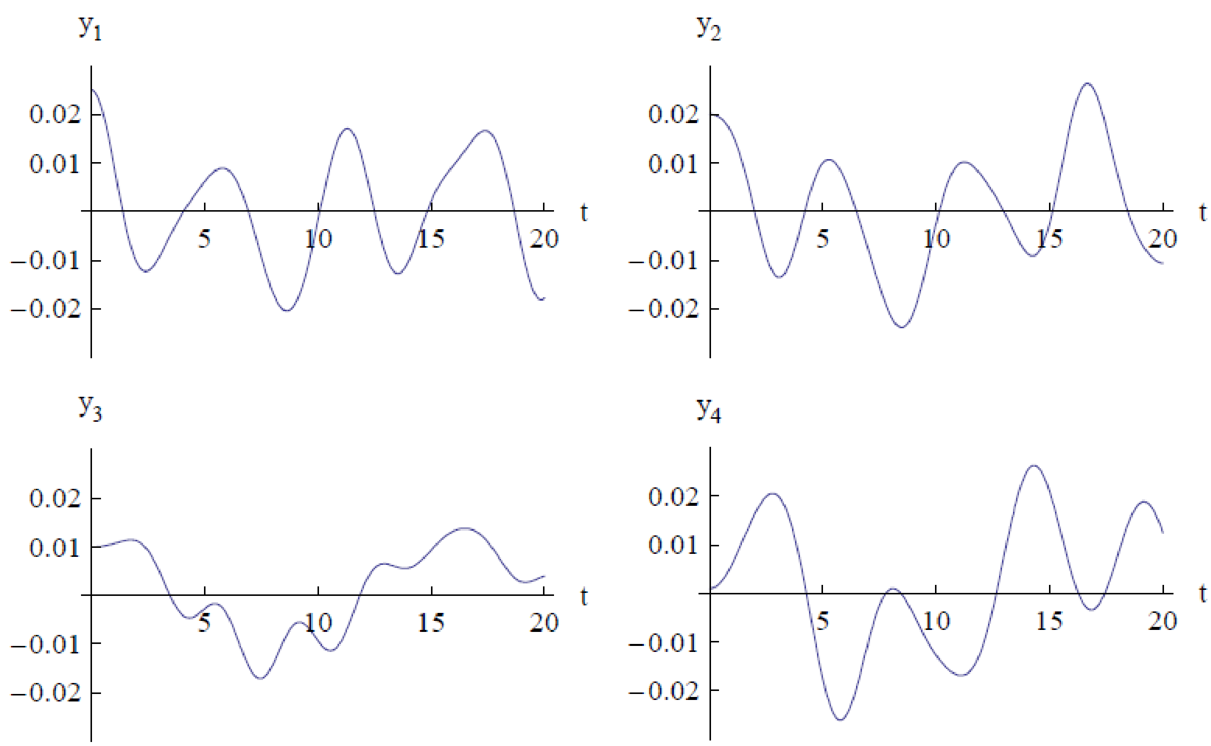
$\ddot{z}_3 + 2.93412\, z_3 = 0;$ $z_3(0) = 0.0130164;$ $\dot{z}_3(0) = 0.$

$\ddot{z}_4 + 4.41511\, z_4 = 0;$ $z_4(0) = -0.00625569;$ $\dot{z}_4(0) = 0.$

b. solution

$$z_1(t) = 0.0414018 \cos(0.388289\, t)$$
$$z_2(t) = 0.0508068 \cos(1.11803\, t)$$
$$z_3(t) = 0.0130164 \cos(1.71293\, t)$$
$$z_4(t) = -0.00625569 \cos(2.10122\, t)$$

$$y_j = \sum z_i\, \phi_i$$



6. Rayleigh damping

- The undamped free vibration mode shapes are orthogonal with respect to the mass and stiffness matrices.

- Generally, the undamped free vibration mode shapes are not orthogonal with respect to the damping matrix.

- Generally, equations of motion for damped systems cannot be uncoupled.

However, we can choose damping matrix to be a linear combination of the mass and stiffness matrices. Then, the mode shapes are orthogonal with respect to the damping matrix, and the equations of motion can be uncoupled.

Damping matrix

$$\boldsymbol{c} = \alpha\, \boldsymbol{m} + \beta\, \boldsymbol{k}$$

Equations of motion

$$\boldsymbol{m}\, \ddot{\boldsymbol{y}}(t) + (\alpha\, \boldsymbol{m} + \beta\, \boldsymbol{k})\, \dot{\boldsymbol{y}}(t) + \boldsymbol{k}\, \boldsymbol{y}(t) = \boldsymbol{f}\,(t); \quad \boldsymbol{y}(0) = \boldsymbol{u}^0 \text{ and } \dot{\boldsymbol{y}}(0) = \boldsymbol{v}^0$$

Displacement vector

$$\boldsymbol{y}\,(t) = \boldsymbol{\Phi}\, \boldsymbol{z}\,(t)$$

where

$$\boldsymbol{\Phi} = (\, \boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \ldots \quad \boldsymbol{\phi}_n\,)\,, \qquad \boldsymbol{z} = (\, z_1 \quad z_2 \quad \ldots \quad z_n\,)^T$$

Uncoupling equations of motion

$$M_i\, \ddot{z}_i(t) + (\alpha\, M_i + \beta\, K_i\,)\, \dot{z}_i(t) + K_i\, z_i(t) = F_i(t); \quad i = 1, 2, \ldots$$

$$z_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{u}^0\right); \quad \dot{z}_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{v}^0\right);$$

where

$$M_i = \boldsymbol{\phi}_i^T\, \boldsymbol{m}\, \boldsymbol{\phi}_i; \quad K_i = \boldsymbol{\phi}_i^T\, \boldsymbol{k}\, \boldsymbol{\phi}_i; \quad \text{and} \quad F_i = \boldsymbol{\phi}_i^T\, \boldsymbol{f}$$

Rewrite the equations of motion

$$\ddot{z}_i(t) + 2\,\xi_i\, \omega_i\, \dot{z}_i(t) + \omega_i^2\, z_i(t) = \frac{1}{M_i}\, F_i(t); \quad i = 1, 2, \ldots$$

where

$$\omega_i = \sqrt{K_i / M_i}$$

$$2\,\xi_i\,\omega_i \equiv \frac{\alpha\,M_i + \beta\,k_i}{M_i} \quad \Longrightarrow \quad \xi_i = \frac{\alpha}{2\,\omega_i} + \frac{\beta\,\omega_i}{2}$$

There are

$$\xi_i = \frac{\alpha}{2\,\omega_i} + \frac{\beta\,\omega_i}{2}$$

$$\xi_j = \frac{\alpha}{2\,\omega_j} + \frac{\beta\,\omega_j}{2}$$

So that

$$\alpha = \frac{2\left(\xi_j\,\omega_i^2\,\omega_j - \xi_i\,\omega_i\,\omega_j^2\right)}{\omega_i^2 - \omega_j^2}; \quad \beta = \frac{2\left(\xi_i\,\omega_i - \xi_j\,\omega_j\right)}{\omega_i^2 - \omega_j^2}$$

Free vibration solution of an undamped system

$$\ddot{z}_i(t) + 2\,\xi_i\,\omega_i\,\dot{z}_i(t) + \omega_i^2\,z_i(t) = 0; \quad i = 1, 2, \ldots$$

$$z_i(t) = e^{-\xi_i\,\omega_i\,t}\left(z_i(0)\cos\omega_{d_i}\,t + \frac{\xi_i\,\omega_i\,z_i(0) + \dot{z}_i(0)}{\omega_{d_i}}\sin\omega_{d_i}\,t\right)$$

$$\omega_{d_i} = \omega_i\sqrt{1 - \xi_i^2}$$

Therefore, the exact solution is

$$\boldsymbol{y}(t) = \sum_i z_i(t)\,\boldsymbol{\phi}_i$$

Approximate solution can be obtained via using the first few mode shapes as usual.

Example 1:

In a four DOF system the damping in the first mode is 0.02 and in the fourth mode is 0.01. Determine the proportional damping matrix and calculate the damping in the second and third modes.

$$\mathbf{m} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}; \qquad \mathbf{k} = \begin{pmatrix} 30 & -7 & 0 & 0 \\ -7 & 20 & -10 & 0 \\ 0 & -10 & 10 & -5 \\ 0 & 0 & -5 & 15 \end{pmatrix}$$

	Eigenvalue	Frequency	Mode shape
1	0.390064	0.624551	(−0.0550491 −0.220587 −0.359618 −0.137788)
2	3.06293	1.75012	(−0.0882154 −0.185068 −0.0249605 0.396667)
3	4.6074	2.14648	(0.243169 0.241884 −0.243678 0.151598)
4	6.93961	2.63431	(0.360633 −0.24204 0.103308 −0.0262229)

Damping in the first mode and fourth mode:

	ω	ξ
1	0.624551	0.02
2	2.63431	0.01

The coefficients in the damping matrix can be determined as

$\alpha = 0.0233321$ $\qquad$ $\beta = 0.00422995$

Damping in other modes:

$$\xi_i = \frac{\alpha}{2\,\omega_i} + \frac{\beta\,\omega_i}{2}$$

	1	2	3	4
ω	0.624551	1.75012	2.14648	2.63431
ξ	0.02	0.0103673	0.00997471	0.01

The damping matrix is

$$c = \begin{pmatrix} 0.243559 & -0.0296096 & 0 & 0 \\ -0.0296096 & 0.201259 & -0.0422995 & 0 \\ 0 & -0.0422995 & 0.15896 & -0.0211497 \\ 0 & 0 & -0.0211497 & 0.18011 \end{pmatrix}$$

Example 2:

Obtain a free vibration solution for a four DOF system using only two modes. Assume 5% damping in the first two modes.

$$\boldsymbol{m} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}; \qquad \boldsymbol{k} = \begin{pmatrix} 10 & -5 & 0 & 0 \\ -5 & 10 & -5 & 0 \\ 0 & -5 & 10 & -5 \\ 0 & 0 & -5 & 5 \end{pmatrix}; \qquad \boldsymbol{f} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{u}^0 = \{0.025, 0.02, 0.01, 0.001\} \qquad \boldsymbol{v}^0 = \{0, 0, 0, 0\}$$

First two modes:

	Frequency (rad/s)	Frequency (Hz)	Mode shape			
1	0.388289	0.0617981	0.114007	0.214263	0.288675	0.328269
2	1.11803	0.177941	-0.288675	-0.288675	-2.35541×10^{-17}	0.288675

$$\xi = \{0.05, 0.05\}; \qquad \omega_d = \{0.387803, 1.11664\}$$

Uncoupling equations of motion

$$\mathrm{M_i} = \phi_i^T\, \boldsymbol{m}\, \phi_i: \qquad \{1, 1\}$$

$$\mathrm{K_i} = \phi_i^T\, \boldsymbol{k}\, \phi_i: \qquad \{0.150768, 1.25\}$$

$F_i = \phi_i^T f$: $\{0., 0.\}$

Modal equations:

$$\ddot{z}_1 + 0.0388289\,\dot{z}_1 + 0.150768\,z_1 = 0; \qquad z_1(0) = 0.0414018; \qquad \dot{z}_1(0) = 0.$$

$$\ddot{z}_2 + 0.111803\,\dot{z}_2 + 1.25\,z_2 = 0; \qquad z_2(0) = -0.0508068; \qquad \dot{z}_2(0) = 0.$$

Solutions:

$$z_1(t) = 0.0414018\,e^{-0.0194145\,t}\cos(0.387803\,t) + 0.00207268\,e^{-0.0194145\,t}\sin(0.387803\,t)$$

$$z_2(t) = -0.0508068\,e^{-0.0559017\,t}\cos(1.11664\,t) - 0.00254352\,e^{-0.0559017\,t}\sin(1.11664\,t)$$

Final solutions:

$y_j = \sum z_i\,\phi_i$