Lecture 4: Undamped Free Vibration

Reading materials: Section 2.1

1. Introduction

- The terminology of "Free Vibration" is used for the study of natural vibration modes in the absence external loading.

- Free vibration solution of multi-degree of freedom systems follows procedure similar to the one used for a single degree of freedom system.

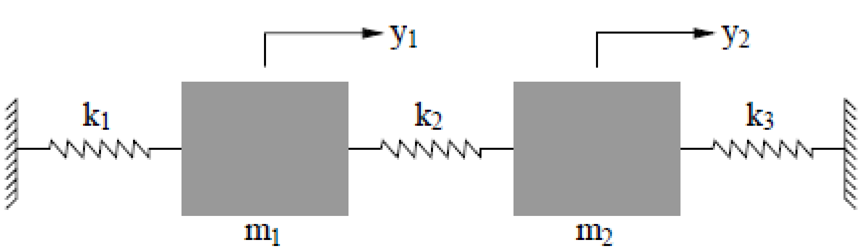
- The number of DOFs of the system is the number of masses in the system multiplying the number of possible types of motion of each mass.

- Generally, the number of equations of motion is the number of DOFs. They are in form of *coupled differential equations*. In other words, each equation involves all the DOFs/coordinates.

- All differential equations for the system must be solved simultaneously.

- The matrix notation is used to indicate the system of equations for a general case.

2. A two DOFs spring-mass system



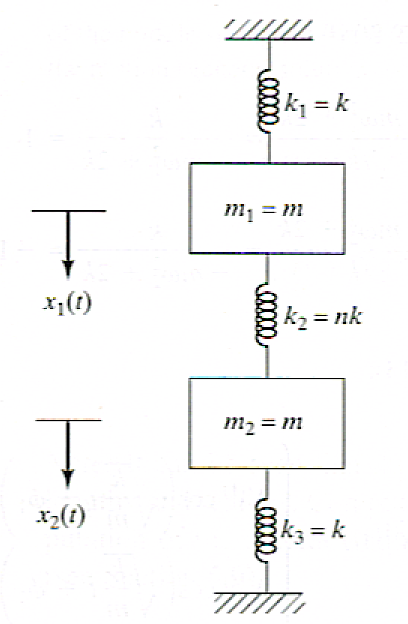
$$\omega_1^2 = \frac{k_2 m_1 + k_3 m_1 + k_1 m_2 + k_2 m_2 - \sqrt{\left(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2\right)^2 - 4\left(k_1 k_2 + k_3 k_2 + k_1 k_3\right) m_1 m_2}}{2 m_1 m_2}$$

$$\omega_2^2 = \frac{k_2 m_1 + k_3 m_1 + k_1 m_2 + k_2 m_2 + \sqrt{\left(-k_2 m_1 - k_3 m_1 - k_1 m_2 - k_2 m_2\right)^2 - 4\left(k_1 k_2 + k_3 k_2 + k_1 k_3\right) m_1 m_2}}{2 m_1 m_2}$$

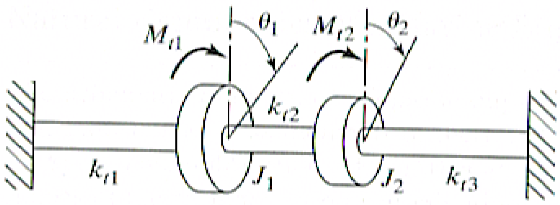
3. Examples:

Compute natural frequencies and mode shapes for a two DOF spring-mass system as shown in above with m_1=9; m_2=1; k_1=38; k_2=2; k_3=3

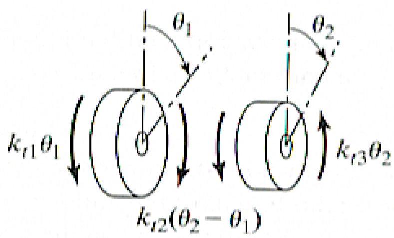
Find the natural frequencies and mode shapes of a spring-mass system, shown below, which is constrained to move in the vertical direction only. (n=1)



4. Torsional System



(a)



(b)

5. Frequencies and mode shapes using standard eigenvalue problem

- If mass matrix is non-singular, the frequency equation can easily be expressed in the form of a standard egienvalue problem.

$$[\boldsymbol{k} - \lambda \boldsymbol{m}]\boldsymbol{\phi} = \left[\boldsymbol{m}^{-1}\boldsymbol{k} - \lambda \boldsymbol{m}^{-1}\boldsymbol{m}\right]\boldsymbol{\phi} = \mathbf{0} \quad \Longrightarrow \quad \left[\boldsymbol{m}^{-1}\boldsymbol{k} - \lambda \boldsymbol{I}\right]\boldsymbol{\phi} = \mathbf{0}$$

- The above is a standard eigenvalue problem. The mode shapes are the eigenvectors while the frequencies are the square roots of the egienvalues.

- Another efficient way for larger systems.

Example: Compute natural frequencies and mode shapes for a two DOF spring-mass system as shown in above with m_1=9; m_2=1; k_1=38; k_2=2; k_3=3

6. Exact solutions based on the given initial conditions

General solutions of the free vibration:

$$y_1(t) = \left(a\, e^{i\,\omega_1 t} + b\, e^{-i\,\omega_1 t}\right)\phi_1^{(1)} + \left(c\, e^{i\,\omega_2 t} + d\, e^{-i\,\omega_2 t}\right)\phi_1^{(2)}$$

$$y_2(t) = \left(a\, e^{i\,\omega_1 t} + b\, e^{-i\,\omega_1 t}\right)\phi_2^{(1)} + \left(c\, e^{i\,\omega_2 t} + d\, e^{-i\,\omega_2 t}\right)\phi_2^{(2)}$$

in matrix form

$$\boldsymbol{y}(t) = \left(a\, e^{i\,\omega_1 t} + b\, e^{-i\,\omega_1 t}\right)\boldsymbol{\phi}^{(1)} + \left(c\, e^{i\,\omega_2 t} + d\, e^{-i\,\omega_2 t}\right)\boldsymbol{\phi}^{(2)}$$

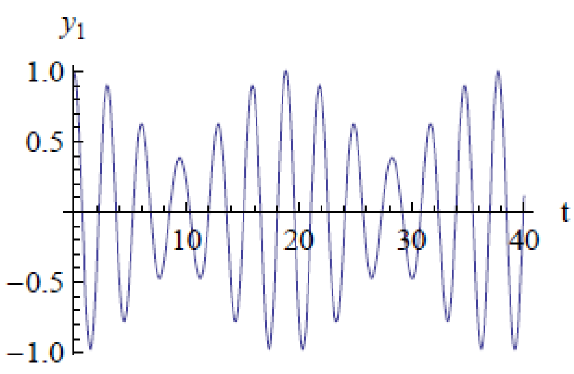
or

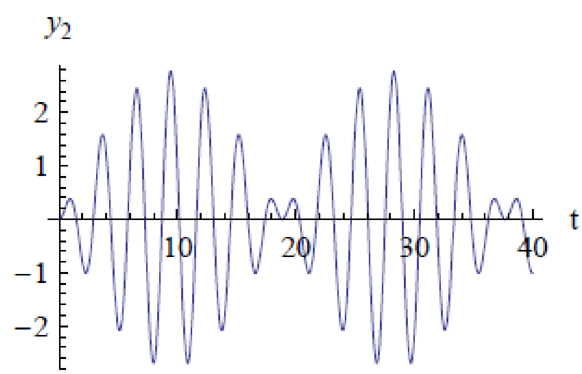
$$\boldsymbol{y}(t) = \sum_m \left(A_m \cos\omega_m t + B_m \sin\omega_m t\right)\boldsymbol{\phi}^{(m)}$$

The unknown coefficients can be determined via initial conditions.

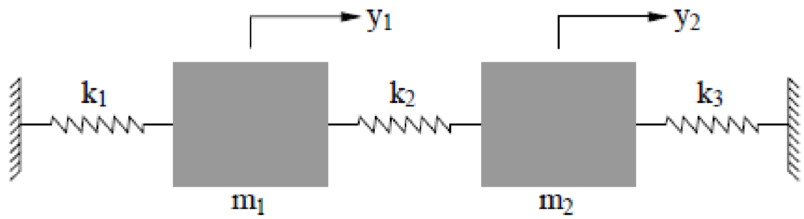
Example:

Compute free vibration solution of a two DOF spring-mass system as shown in above with m_1=9; m_2=1; k_1=38; k_2=2; k_3=3 and the following initial conditions:
$y_1(0) = 1$; $y_2(0) = 0$; $v_1(0) = 0$; $v_2(0) = 0$





7. Interpretation of mode shapes



$$\phi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \quad \phi^{(2)} = \begin{pmatrix} 1 \\ -9/2 \end{pmatrix}$$

- In the first mode shape the two masses are moving in the same direction.

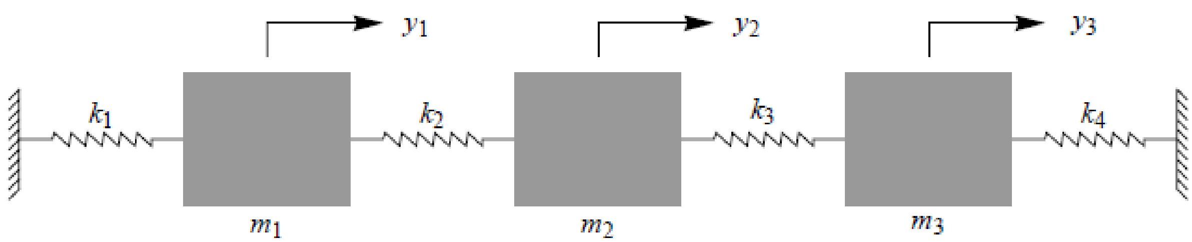
- The movement of the first mass is smaller than the second mass in this mode because the first mass is attached to a stiffer spring.

- In the second mode shape the two masses are moving in the opposite directions. The movement of the first mass is also smaller than the second mass.

- There are only two possibilities of independent motion of this two DOF system indicated by those two mode shapes. Any other motion can be described in terms of a linear combination of these two modes. For example:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{9}{13}\phi^{(1)} + \frac{4}{13}\phi^{(2)}$$

8. Example: free vibration solution of the following three DOF system (Optional)



$$m_1 = m_2 = m_3 = 4; \quad k_1 = k_2 = k_3 = 5; \quad k_4 = 10$$

$$\begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} \dot{y}_1(0) \\ \dot{y}_2(0) \\ \dot{y}_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\omega_1 = \sqrt{\lambda_1} = 0.970194; \qquad \omega_2 = \sqrt{\lambda_2} = 1.74823; \qquad \omega_3 = \sqrt{\lambda_3} = 2.18001$$

$$\boldsymbol{\phi}^{(1)} = \boldsymbol{U}^{-1}\overline{\boldsymbol{\phi}}^{(1)} = \begin{pmatrix} 0.295505 \\ 0.368488 \\ 0.163993 \end{pmatrix}; \quad \boldsymbol{\phi}^{(2)} = \boldsymbol{U}^{-1}\overline{\boldsymbol{\phi}}^{(2)} = \begin{pmatrix} 0.368488 \\ -0.163993 \\ -0.295505 \end{pmatrix};$$

$$\boldsymbol{\phi}^{(3)} = \boldsymbol{U}^{-1}\overline{\boldsymbol{\phi}}^{(3)} = \begin{pmatrix} 0.163993 \\ -0.295505 \\ 0.368488 \end{pmatrix}$$