Lecture 25: Large scale systems

Reading materials: 10.1 and 10.2

1. Guyan Reduction

- Finite element discretization results in a large dynamic system. Therefore, computation is intensive.

- One approach is reducing the size of the eigenvalue problem that must be solved to compute the mode shapes and frequencies.

- Generalized eigenvalue problem

$$\boldsymbol{k}\,\boldsymbol{\phi} = \lambda\,\boldsymbol{m}\,\boldsymbol{\phi}$$

- In the reduction process, choosing an appropriate set of DOFs that are to be retained. Those DOFs are called master DOFs while the ones eliminated are called slave DOFs.

Relationship between the total DOFs (#n) and the master DOFs (#m)s

$$\boldsymbol{\phi} = \boldsymbol{Z}\,\boldsymbol{\psi}$$

$$\boldsymbol{Z}^T\,\boldsymbol{k}\,\boldsymbol{Z}\,\boldsymbol{\psi} = \omega^2\,\boldsymbol{Z}^T\,\boldsymbol{m}\,\boldsymbol{Z}\,\boldsymbol{\psi} \quad \Longrightarrow \quad \overline{\boldsymbol{k}}\,\boldsymbol{\psi} = \omega^2\,\overline{\boldsymbol{m}}\,\boldsymbol{\psi}$$

- Static equilibrium equations

$$\boldsymbol{k}\,\boldsymbol{\phi} = \boldsymbol{r} \quad \Longrightarrow \quad \begin{pmatrix} \boldsymbol{k}_{\text{mm}} & \boldsymbol{k}_{\text{ms}} \\ \boldsymbol{k}_{\text{sm}} & \boldsymbol{k}_{\text{ss}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{0} \end{pmatrix}$$

$$\boldsymbol{k}_{\text{sm}}\,\boldsymbol{\psi} + \boldsymbol{k}_{\text{ss}}\,\boldsymbol{\psi}_s = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{\psi}_s = -\boldsymbol{k}_{\text{ss}}^{-1}\,\boldsymbol{k}_{\text{sm}}\,\boldsymbol{\psi}$$

$$\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\psi}_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{\psi} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm}\,\boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}\boldsymbol{\psi} \quad \Longrightarrow \quad \boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}$$

$$\overline{\boldsymbol{k}} = \boldsymbol{Z}^T\,\boldsymbol{k}\,\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{k}_{mm} & \boldsymbol{k}_{ms} \\ \boldsymbol{k}_{sm} & \boldsymbol{k}_{ss} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}$$

$$\overline{\boldsymbol{m}} = \boldsymbol{Z}^T\,\boldsymbol{m}\,\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}^T \begin{pmatrix} \boldsymbol{m}_{mm} & \boldsymbol{m}_{ms} \\ \boldsymbol{m}_{sm} & \boldsymbol{m}_{ss} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} \\ -\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} \end{pmatrix}$$

$$\overline{\boldsymbol{m}} = \boldsymbol{m}_{mm} - \boldsymbol{k}_{ms}\,\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{m}_{sm} - \boldsymbol{m}_{ms}\,\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm} + \boldsymbol{k}_{ms}\,\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{m}_{ss}\,\boldsymbol{k}_{ss}^{-1}\,\boldsymbol{k}_{sm}$$

Example 1

$$\mathbf{k} = \begin{pmatrix} 8 & -2 & 0 & 0 \\ -2 & 12 & -3 & 0 \\ 0 & -3 & 16 & -4 \\ 0 & 0 & -4 & 4 \end{pmatrix}; \qquad \mathbf{m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Master dof = {2, 4}; Slave dof = {1, 3}

$$\boldsymbol{k}_{mm} = \begin{pmatrix} 12 & 0 \\ 0 & 4 \end{pmatrix}; \qquad \boldsymbol{k}_{ms} = \begin{pmatrix} -2 & -3 \\ 0 & -4 \end{pmatrix};$$

$$\boldsymbol{k}_{sm} = \begin{pmatrix} -2 & 0 \\ -3 & -4 \end{pmatrix}; \qquad \boldsymbol{k}_{ss} = \begin{pmatrix} 8 & 0 \\ 0 & 16 \end{pmatrix}$$

$$m_{mm} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \qquad m_{ms} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

$$m_{sm} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \qquad m_{ss} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$k_{ss}^{-1} = \begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{16} \end{pmatrix} \qquad k_{ss}^{-1} k_{sm} = \begin{pmatrix} -\frac{1}{4} & 0 \\ -\frac{3}{16} & -\frac{1}{4} \end{pmatrix}$$

Z matrix with the order of dof as {2, 4, 1, 3}

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{4} & 0 \\ \frac{3}{16} & \frac{1}{4} \end{pmatrix}$$

In the standard order of dof {1, 2, 3, 4}

$$Z = \begin{pmatrix} \frac{1}{4} & 0 \\ 1 & 0 \\ \frac{3}{16} & \frac{1}{4} \\ 0 & 1 \end{pmatrix}$$

Reduced matrices

$$\bar{k} = \begin{pmatrix} \frac{175}{16} & -\frac{3}{4} \\ -\frac{3}{4} & 3 \end{pmatrix} \qquad \bar{m} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$\lambda_1 = 2.8909; \qquad \Psi_1 = \{0.143955,\ 0.989584\}$

$\lambda_2 = 5.57785; \qquad \Psi_2 = \{0.960187,\ -0.279357\}$

For original problem

$\lambda_1 = 2.8909; \qquad \boldsymbol{\phi}_1 = \boldsymbol{Z}\boldsymbol{\psi}_1 = \{0.0359887,\ 0.143955,\ 0.274388,\ 0.989584\}$

$\lambda_2 = 5.57785; \qquad \boldsymbol{\phi}_2 = \boldsymbol{Z}\boldsymbol{\psi}_2 = \{0.240047,\ 0.960187,\ 0.110196,\ -0.279357\}$

	Frequency (rad/s)	Frequency (Hz)	Mode shape			
1	1.70026	0.270605	0.0359887	0.143955	0.274388	0.989584
2	2.36175	0.375884	0.240047	0.960187	0.110196	−0.279357

2. Inverse iteration

An iterative method to compute frequencies and modes shapes for multi-degree freedom systems.

$$\left[\boldsymbol{k} - \omega^2 \boldsymbol{m}\right]\boldsymbol{\phi} = \boldsymbol{0}$$

Rearrange

$$\boldsymbol{\phi} = \omega^2 \boldsymbol{k}^{-1} \boldsymbol{m}\,\boldsymbol{\phi} \equiv \omega^2 \boldsymbol{D}\,\boldsymbol{\phi}$$

Dynamic matrix

$$\boldsymbol{D} = \boldsymbol{k}^{-1}\boldsymbol{m}$$

$$\bar{z}_{i+1} = \boldsymbol{D}\, z_i$$

$$z_{i+1} = \frac{\bar{z}_{i+1}}{\sqrt{\bar{z}_{i+1}^T\, \bar{z}_{i+1}}}; \quad i = 0,\, 1,\, \ldots$$

$$z = \omega^2\, \boldsymbol{D}\, z \implies z^T z = \omega^2\, z^T \boldsymbol{D}\, z \implies \omega^2 = \frac{z^T z}{z^T \boldsymbol{D}\, z}$$

Example 2

$$\boldsymbol{k} = \begin{pmatrix} 137.78 & -111.88 & 0 \\ -111.88 & 286.7 & -174.82 \\ 0 & -174.82 & 174.82 \end{pmatrix}$$

$$\boldsymbol{m} = \begin{pmatrix} 0.05823 & 0 & 0 \\ 0 & 0.04658 & 0 \\ 0 & 0 & 0.03494 \end{pmatrix}$$

$$\boldsymbol{k}^{-1} = \begin{pmatrix} 0.03861 & 0.03861 & 0.03861 \\ 0.03861 & 0.0475482 & 0.0475482 \\ 0.03861 & 0.0475482 & 0.0532684 \end{pmatrix}$$

$$\boldsymbol{D} = \boldsymbol{k}^{-1}\boldsymbol{m} = \begin{pmatrix} 0.00224826 & 0.00179846 & 0.00134903 \\ 0.00224826 & 0.00221479 & 0.00166133 \\ 0.00224826 & 0.00221479 & 0.0018612 \end{pmatrix}$$

$$z_0 = \begin{pmatrix} 1. \\ 1. \\ 1. \end{pmatrix}; \qquad \bar{z}_1 = \boldsymbol{D} z_0 = \begin{pmatrix} 0.00539575 \\ 0.00612439 \\ 0.00632425 \end{pmatrix}; \qquad z_1 = \bar{z}_1/\mathrm{Sqrt}[\bar{z}_1^T\, \bar{z}_1] = \begin{pmatrix} 0.52256 \\ 0.593126 \\ 0.612482 \end{pmatrix}$$

$$z_1 = \begin{pmatrix} 0.52256 \\ 0.593126 \\ 0.612482 \end{pmatrix}; \quad \bar{z}_2 = \boldsymbol{D} z_1 = \begin{pmatrix} 0.00306782 \\ 0.00350604 \\ 0.00362845 \end{pmatrix}; \quad z_2 = \bar{z}_2/\mathrm{Sqrt}[\bar{z}_2^{\mathrm{T}} \bar{z}_2] = \begin{pmatrix} 0.519526 \\ 0.593737 \\ 0.614467 \end{pmatrix}$$

$$z_2 = \begin{pmatrix} 0.519526 \\ 0.593737 \\ 0.614467 \end{pmatrix}; \quad \bar{z}_3 = \boldsymbol{D} z_2 = \begin{pmatrix} 0.00306478 \\ 0.00350387 \\ 0.00362668 \end{pmatrix}; \quad z_3 = \bar{z}_3/\mathrm{Sqrt}[\bar{z}_3^{\mathrm{T}} \bar{z}_3] = \begin{pmatrix} 0.519359 \\ 0.593767 \\ 0.614579 \end{pmatrix}$$

$$z_3 = \begin{pmatrix} 0.519359 \\ 0.593767 \\ 0.614579 \end{pmatrix}; \quad \bar{z}_4 = \boldsymbol{D} z_3 = \begin{pmatrix} 0.00306461 \\ 0.00350375 \\ 0.00362658 \end{pmatrix}; \quad z_4 = \bar{z}_4/\mathrm{Sqrt}[\bar{z}_4^{\mathrm{T}} \bar{z}_4] = \begin{pmatrix} 0.519349 \\ 0.593769 \\ 0.614585 \end{pmatrix}$$

$\omega_1 = 13.018$ rad/s; $\phi_1^{\mathrm{T}} = \{0.519349, 0.593769, 0.614585\}$

	Frequency (rad/s)	Frequency (Hz)	Mode shape		
1	13.018	2.07187	0.519349	0.593769	0.614585