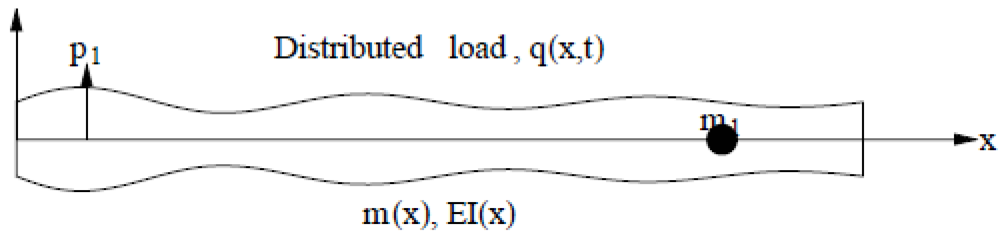
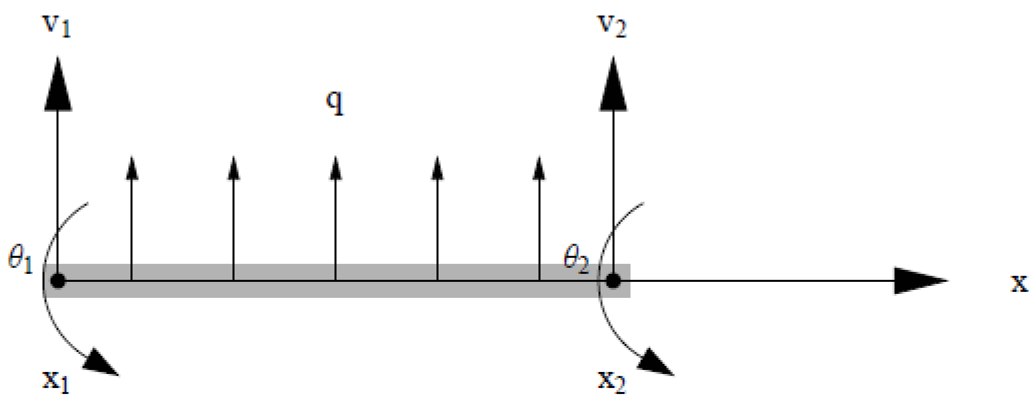
# Lecture 24: Finite element method: Beams and Frames

Reading materials: Section 9.3

## 1. Beam element




Nodal variables

Nodal loads: vertical loads and moments

Nodal degree of freedom: vertical displacements and rotations

$$v(x,t) = \sum_{i=1}^{4} N_i(x)\, d_i(t) \equiv \boldsymbol{N}^T \boldsymbol{d}$$

$$\boldsymbol{d} = (\, v_1 \quad \theta_1 \quad v_2 \quad \theta_2 \,)^T$$

$$v(x) = a_3\, x^3 + a_2\, x^2 + a_1\, x + a_0$$
$$v(0) = v_1 \Longrightarrow a_0 = v_1$$
$$v(L) = v_2 \Longrightarrow a_3\, L^3 + a_2\, L^2 + a_1\, L + a_0 = v_2$$
$$\theta(0) = \theta_1 \Longrightarrow a_1 = \theta_1$$
$$\theta(L) = \theta_2 \Longrightarrow 3\, a_3\, L^2 + 2\, a_2\, L + a_1 = \theta_2$$

$$N^{\mathrm{T}} = \left\{\frac{2\,x^3}{L^3} - \frac{3\,x^2}{L^2} + 1,\ \frac{x^3}{L^2} - \frac{2\,x^2}{L} + x,\ \frac{3\,x^2}{L^2} - \frac{2\,x^3}{L^3},\ \frac{x^3}{L^2} - \frac{x^2}{L}\right\}$$

- Potential

$$U_s = \frac{1}{2}\int_0^L \mathrm{EI}\left(\frac{d^2\, v(x,t)}{\mathrm{dx}^2}\right)^2 \mathrm{dx} = \frac{1}{2}\,\boldsymbol{d}^T \int_0^L \mathrm{EI}\,\boldsymbol{B}\,\boldsymbol{B}^T\,\mathrm{dx}\,\boldsymbol{d} \equiv \frac{1}{2}\,\boldsymbol{d}^T\,\boldsymbol{k}\,\boldsymbol{d}$$

$$\frac{d^2\, v(x,t)}{\mathrm{dx}^2} = \sum_{i=1}^4 \frac{d^2\, N_i}{\mathrm{dx}^2}\, d_i(t) \equiv \boldsymbol{B}^T\,\boldsymbol{d}$$
$$\boldsymbol{B}^{\mathrm{T}} = \left\{\frac{12\,x}{L^3} - \frac{6}{L^2},\ \frac{6\,x}{L^2} - \frac{4}{L},\ \frac{6}{L^2} - \frac{12\,x}{L^3},\ \frac{6\,x}{L^2} - \frac{2}{L}\right\}$$

$$\boldsymbol{k} = \int_0^L \mathrm{EI}\,\boldsymbol{B}\,\boldsymbol{B}^T\,\mathrm{dx}$$

- Kinetic energy

Kinetic energy due to the distributed mass

$$T_d = \frac{1}{2}\int_0^L m\left(\frac{\mathrm{dy}(x,t)}{\mathrm{dt}}\right)^2 \mathrm{dx} = \frac{1}{2}\,\dot{\boldsymbol{u}}^T \int_0^L m\,\boldsymbol{N}\,\boldsymbol{N}^T\,\mathrm{dx}\,\dot{\boldsymbol{u}} \equiv \frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}_d\,\dot{\boldsymbol{u}}$$

Kinetic energy due to the concentrated masses at nodes

$$T_c = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2\right) = \frac{1}{2}\dot{\boldsymbol{d}}^T \boldsymbol{m}_c \dot{\boldsymbol{d}}$$

$$T = T_d + T_c = \frac{1}{2}\dot{\boldsymbol{d}}^T (\boldsymbol{m}_d + \boldsymbol{m}_c)\dot{\boldsymbol{d}} \equiv \frac{1}{2}\dot{\boldsymbol{d}}^T \boldsymbol{m}\dot{\boldsymbol{d}}$$

$$\boldsymbol{m} = \boldsymbol{m}_d + \boldsymbol{m}_c = \begin{pmatrix} \frac{13\,L\,m}{35} + m_1 & \frac{11\,L^2\,m}{210} & \frac{9\,L\,m}{70} & -\frac{13\,L^2\,m}{420} \\ \frac{11\,L^2\,m}{210} & \frac{L^3\,m}{105} & \frac{13\,L^2\,m}{420} & -\frac{L^3\,m}{140} \\ \frac{9\,L\,m}{70} & \frac{13\,L^2\,m}{420} & \frac{13\,L\,m}{35} + m_2 & -\frac{11\,L^2\,m}{210} \\ -\frac{13\,L^2\,m}{420} & -\frac{L^3\,m}{140} & -\frac{11\,L^2\,m}{210} & \frac{L^3\,m}{105} \end{pmatrix}$$

work done by external loads

work done by the distributed load

$$W_q = \int_0^L q(t)\, v(x, t)\, \mathrm{d}x = \int_0^L q\, \boldsymbol{N}^T\, \mathrm{d}x\, \boldsymbol{u} \equiv \boldsymbol{r}_q^T \boldsymbol{u}$$

work done by the concentrated loads and moments

$$W_p = (F_1 v_1 + F_2 v_2 + M_1 \theta_1 + M_2 \theta_2) = r_p^T \boldsymbol{d}$$

$$\boldsymbol{r}_p = (\,F_1 \quad M_1 \quad F_2 \quad M_2\,)^T$$

$$W = W_q + W_p = \left(\boldsymbol{r}_q + \boldsymbol{r}_p\right)^T \boldsymbol{d} \equiv \boldsymbol{r}^T \boldsymbol{d}$$

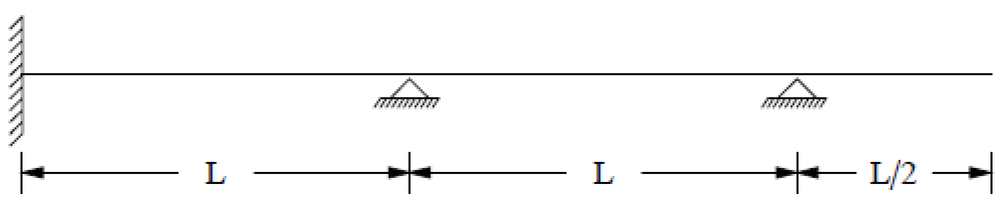
$$\boldsymbol{r} = \boldsymbol{r}_q + \boldsymbol{r}_p = \begin{pmatrix} \frac{L\,q}{2} + F_1 \\ \frac{L^2\,q}{12} + M_1 \\ \frac{L\,q}{2} + F_2 \\ -\frac{L^2\,q}{12} + M_2 \end{pmatrix}$$

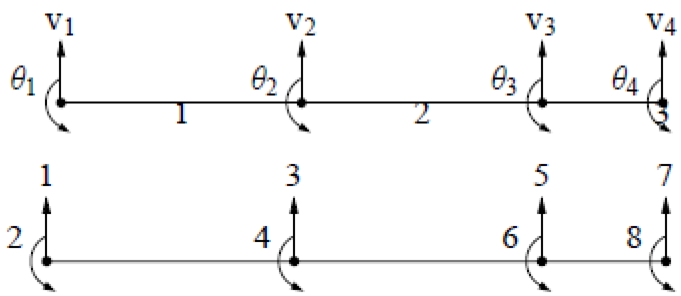
- Equations of motion

$$\frac{d}{dt}(T+U) = 0 \quad \Longrightarrow \quad \frac{d}{dt}\left[\frac{1}{2}\,\dot{\boldsymbol{d}}^T\,\boldsymbol{m}\,\dot{\boldsymbol{d}} + \frac{1}{2}\,\boldsymbol{d}^T\,\boldsymbol{k}\,\boldsymbol{d} - \boldsymbol{r}^T\,\boldsymbol{d}\right] = \dot{\boldsymbol{d}}^T\left[\boldsymbol{m}\,\ddot{\boldsymbol{d}} + \boldsymbol{k}\,\boldsymbol{d} - \boldsymbol{r}\right] = 0$$

$$\boldsymbol{m}\,\ddot{\boldsymbol{d}}(t) + \boldsymbol{k}\,\boldsymbol{d}(t) = \boldsymbol{r}(t)$$

- Example



## Element 1

$$\text{Element vectors contribute to } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and matrices to } \begin{pmatrix} [1,1] & [1,2] & [1,3] & [1,4] \\ [2,1] & [2,2] & [2,3] & [2,4] \\ [3,1] & [3,2] & [3,3] & [3,4] \\ [4,1] & [4,2] & [4,3] & [4,4] \end{pmatrix}$$

$$\begin{pmatrix} 0.00156 & 0.022 & 0.00054 & -0.013 & 0 & 0 & 0 & 0 \\ 0.022 & 0.4 & 0.013 & -0.3 & 0 & 0 & 0 & 0 \\ 0.00054 & 0.013 & 0.00156 & -0.022 & 0 & 0 & 0 & 0 \\ -0.013 & -0.3 & -0.022 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \\ \ddot{w}_3 \\ \ddot{\theta}_3 \\ \ddot{w}_4 \\ \ddot{\theta}_4 \end{pmatrix} + \begin{pmatrix} 120 & 6000 & -120 & 6000 & 0 & 0 & 0 & 0 \\ 6000 & 400000 & -6000 & 200000 & 0 & 0 & 0 & 0 \\ -120 & -6000 & 120 & -6000 & 0 & 0 & 0 & 0 \\ 6000 & 200000 & -6000 & 400000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## Element 2

$$\text{Element vectors contribute to } \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \text{ and matrices to } \begin{pmatrix} [3,3] & [3,4] & [3,5] & [3,6] \\ [4,3] & [4,4] & [4,5] & [4,6] \\ [5,3] & [5,4] & [5,5] & [5,6] \\ [6,3] & [6,4] & [6,5] & [6,6] \end{pmatrix}$$

$$\begin{pmatrix} 0.00156 & 0.022 & 0.00054 & -0.013 & 0 & 0 & 0 & 0 \\ 0.022 & 0.4 & 0.013 & -0.3 & 0 & 0 & 0 & 0 \\ 0.00054 & 0.013 & 0.00312 & 0 & 0.00054 & -0.013 & 0 & 0 \\ -0.013 & -0.3 & 0 & 0.8 & 0.013 & -0.3 & 0 & 0 \\ 0 & 0 & 0.00054 & 0.013 & 0.00156 & -0.022 & 0 & 0 \\ 0 & 0 & -0.013 & -0.3 & -0.022 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \\ \ddot{w}_3 \\ \ddot{\theta}_3 \\ \ddot{w}_4 \\ \ddot{\theta}_4 \end{pmatrix} + \begin{pmatrix} 120 & 6000 & -120 & 6000 & 0 & 0 & 0 & 0 \\ 6000 & 400000 & -6000 & 200000 & 0 & 0 & 0 & 0 \\ -120 & -6000 & 240 & 0 & -120 & 6000 & 0 & 0 \\ 6000 & 200000 & 0 & 800000 & -6000 & 200000 & 0 & 0 \\ 0 & 0 & -120 & -6000 & 120 & -6000 & 0 & 0 \\ 0 & 0 & 6000 & 200000 & -6000 & 400000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## Element 3

$$\text{Element vectors contribute to} \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \text{and matrices to} \begin{pmatrix} [5,5] & [5,6] & [5,7] & [5,8] \\ [6,5] & [6,6] & [6,7] & [6,8] \\ [7,5] & [7,6] & [7,7] & [7,8] \\ [8,5] & [8,6] & [8,7] & [8,8] \end{pmatrix}$$

$$\begin{pmatrix} 0.00156 & 0.022 & 0.00054 & -0.013 & 0 & 0 & 0 & 0 \\ 0.022 & 0.4 & 0.013 & -0.3 & 0 & 0 & 0 & 0 \\ 0.00054 & 0.013 & 0.00312 & 0 & 0.00054 & -0.013 & 0 & 0 \\ -0.013 & -0.3 & 0 & 0.8 & 0.013 & -0.3 & 0 & 0 \\ 0 & 0 & 0.00054 & 0.013 & 0.00234 & -0.0165 & 0.00027 & -0.00325 \\ 0 & 0 & -0.013 & -0.3 & -0.0165 & 0.45 & 0.00325 & -0.0375 \\ 0 & 0 & 0 & 0 & 0.00027 & 0.00325 & 0.00078 & -0.0055 \\ 0 & 0 & 0 & 0 & -0.00325 & -0.0375 & -0.0055 & 0.05 \end{pmatrix} \cdot \begin{pmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \\ \ddot{w}_3 \\ \ddot{\theta}_3 \\ \ddot{w}_4 \\ \ddot{\theta}_4 \end{pmatrix}$$

$$+ \begin{pmatrix} 120 & 6000 & -120 & 6000 & 0 & 0 & 0 & 0 \\ 6000 & 400000 & -6000 & 200000 & 0 & 0 & 0 & 0 \\ -120 & -6000 & 240 & 0 & -120 & 6000 & 0 & 0 \\ 6000 & 200000 & 0 & 800000 & -6000 & 200000 & 0 & 0 \\ 0 & 0 & -120 & -6000 & 1080. & 18000. & -960. & 24000. \\ 0 & 0 & 6000 & 200000 & 18000. & 1.2\times 10^6 & -24000. & 400000. \\ 0 & 0 & 0 & 0 & -960. & -24000. & 960. & -24000. \\ 0 & 0 & 0 & 0 & 24000. & 400000. & -24000. & 800000. \end{pmatrix} \begin{pmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

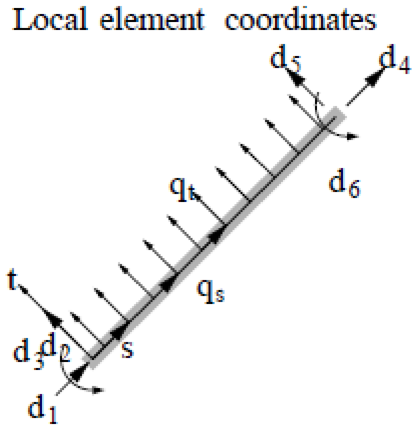
Boundary conditions

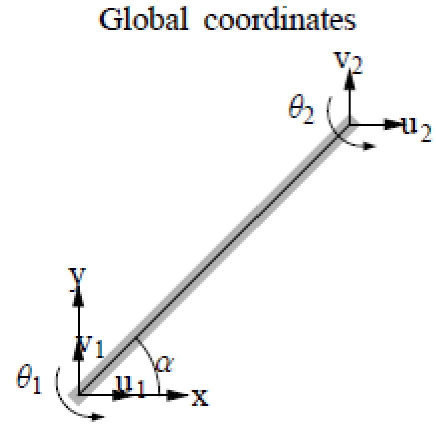
| Node | dof | Value |
|---|---|---|
| 1 | $w_1$ | 0 |
| | $\theta_1$ | 0 |
| 2 | $w_2$ | 0 |
| 3 | $w_3$ | 0 |

Remove {1, 2, 3, 5} rows and columns.

$$\begin{pmatrix} 0.8 & -0.3 & 0 & 0 \\ -0.3 & 0.45 & 0.00325 & -0.0375 \\ 0 & 0.00325 & 0.00078 & -0.0055 \\ 0 & -0.0375 & -0.0055 & 0.05 \end{pmatrix} \begin{pmatrix} \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{w}_4 \\ \ddot{\theta}_4 \end{pmatrix} + \begin{pmatrix} 800000 & 200000 & 0 & 0 \\ 200000 & 1.2\times 10^6 & -24000. & 400000. \\ 0 & -24000. & 960. & -24000. \\ 0 & 400000. & -24000. & 800000. \end{pmatrix} \begin{pmatrix} \theta_2 \\ \theta_3 \\ w_4 \\ \theta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## 2. Plane Frame element




$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} \ell_s & m_s & 0 & 0 & 0 \\ -m_s & \ell_s & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \ell_s & m_s & 0 \\ 0 & 0 & -m_s & \ell_s & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{pmatrix} \Rightarrow \quad \boldsymbol{d_\ell} = \boldsymbol{T}\,\boldsymbol{d}$$

$$\boldsymbol{d} = \boldsymbol{T}^T\,\boldsymbol{d_\ell}$$

$$\boldsymbol{m} = \boldsymbol{T}^T\,\boldsymbol{m_\ell}\,\boldsymbol{T}$$

$$\boldsymbol{k} = \boldsymbol{T}^T\,\boldsymbol{k_\ell}\,\boldsymbol{T}$$

$$\boldsymbol{r} = \boldsymbol{T}^T\,\boldsymbol{r_\ell}$$