Lecture 23: Finite element method: vibrations of trusses
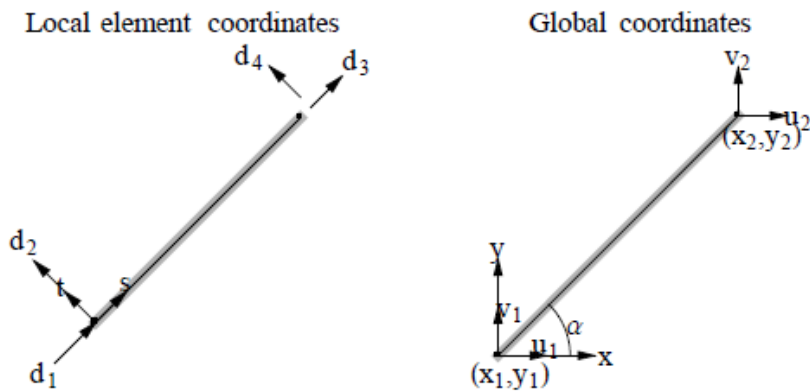
Reading materials: Section 9.2

1. Plane truss element

Local element coordinates   Global coordinates

🔵 In terms of local coordinates

$$m_\ell \ddot{d}_\ell + k_\ell d_\ell = 0$$

or

$$\frac{L m}{6}\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} \ddot{d}_1 \\ \ddot{d}_2 \\ \ddot{d}_3 \\ \ddot{d}_4 \end{pmatrix} + \frac{EA}{L}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Assume that there is no distributed force.

Degree of freedom

$$d = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix};$$

Nodal forces

$$r = \begin{pmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{pmatrix}$$

● In terms of global coordinates

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} \ell_s & m_s & 0 & 0 \\ -m_s & \ell_s & 0 & 0 \\ 0 & 0 & \ell_s & m_s \\ 0 & 0 & -m_s & \ell_s \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \Rightarrow d_\ell = T d$$

$$\ell_s = \cos \theta = \frac{x_2 - x_1}{L} \quad m_s = \sin\theta = \frac{y_2 - y_1}{L}$$

where

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

● Equations of motion

$$d = T^T d_\ell$$

$$m_\ell \, \ddot{d}_\ell + k_\ell \, d_\ell = 0 \quad \Longrightarrow \quad m_\ell \, T \, \ddot{d} + k_\ell \, T \, d = 0$$

$$T^T \, m_\ell \, T \, \ddot{d} + T^T \, k_\ell \, T \, d = 0$$

Finally,

$$m \, \ddot{d} + k \, d = 0$$

where

$$m = T^T \, m_\ell \, T \quad \text{and} \quad k = T^T \, k_\ell \, T$$

Or

$$\mathbf{m} = T^T m_\ell T = \begin{pmatrix} \dfrac{Lm}{3} & 0 & \dfrac{Lm}{6} & 0 \\[2mm] 0 & \dfrac{Lm}{3} & 0 & \dfrac{Lm}{6} \\[2mm] \dfrac{Lm}{6} & 0 & \dfrac{Lm}{3} & 0 \\[2mm] 0 & \dfrac{Lm}{6} & 0 & \dfrac{Lm}{3} \end{pmatrix}$$

$$\mathbf{k} = T^T k_\ell T = \dfrac{EA}{L} \begin{pmatrix} \ell_s^2 & m_s\,\ell_s & -\ell_s^2 & -m_s\,\ell_s \\[2mm] m_s\,\ell_s & m_s^2 & -m_s\,\ell_s & -m_s^2 \\[2mm] -\ell_s^2 & -m_s\,\ell_s & \ell_s^2 & m_s\,\ell_s \\[2mm] -m_s\,\ell_s & -m_s^2 & m_s\,\ell_s & m_s^2 \end{pmatrix}$$

Example
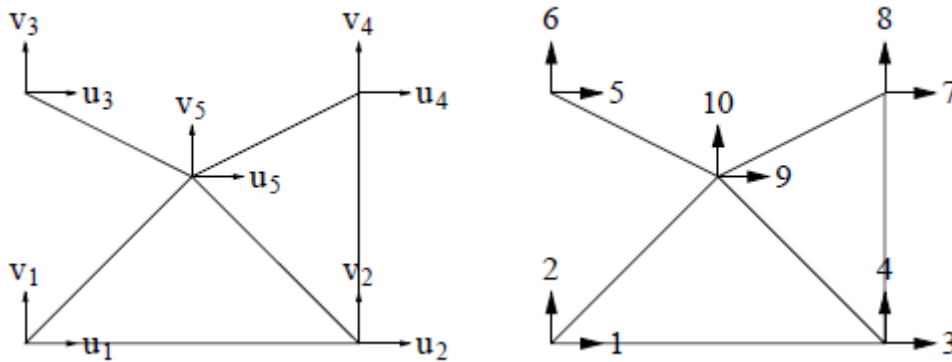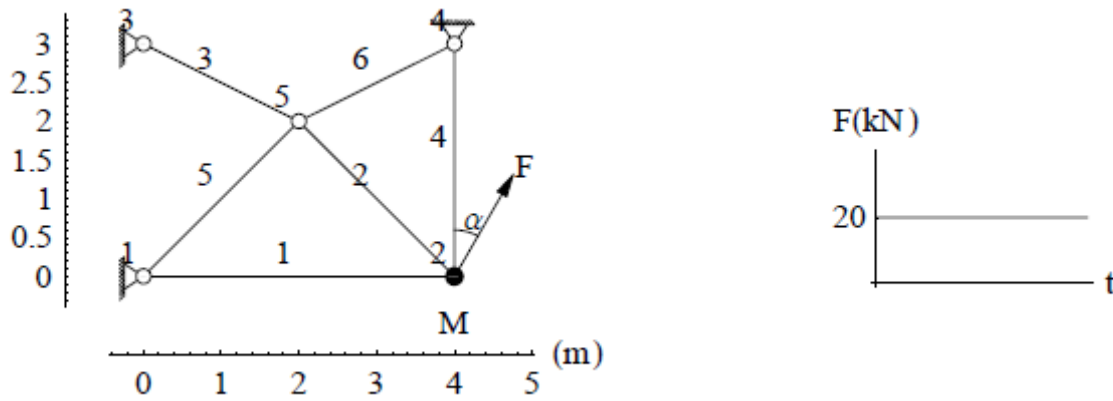
Element 1

$$
\begin{pmatrix}
10.48 & 0 & 5.24 & 0 \\
0 & 10.48 & 0 & 5.24 \\
5.24 & 0 & 10.48 & 0 \\
0 & 5.24 & 0 & 10.48
\end{pmatrix}
\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}
+
\begin{pmatrix}
52500. & 0. & -52500. & 0. \\
0. & 0. & 0. & 0. \\
-52500. & 0. & 52500. & 0. \\
0. & 0. & 0. & 0.
\end{pmatrix}
\begin{pmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$\text{Element vectors contribute to } \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ and matrices to } \begin{pmatrix} [1,1] & [1,2] & [1,3] & [1,4] \\ [2,1] & [2,2] & [2,3] & [2,4] \\ [3,1] & [3,2] & [3,3] & [3,4] \\ [4,1] & [4,2] & [4,3] & [4,4] \end{pmatrix}$$

$$\begin{pmatrix}
10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0 \\
5.24 & 0 & 110.48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5.24 & 0 & 110.48 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix}
52500. & 0 & -52500. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-52500. & 0 & 52500. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Element 2

$$\begin{pmatrix}
7.41048 & 0 & 3.70524 & 0 \\
0 & 7.41048 & 0 & 3.70524 \\
3.70524 & 0 & 7.41048 & 0 \\
0 & 3.70524 & 0 & 7.41048
\end{pmatrix}
\begin{pmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{u}_5 \\ \ddot{v}_5 \end{pmatrix}
+
\begin{pmatrix}
37123.1 & -37123.1 & -37123.1 & 37123.1 \\
-37123.1 & 37123.1 & 37123.1 & -37123.1 \\
-37123.1 & 37123.1 & 37123.1 & -37123.1 \\
37123.1 & -37123.1 & -37123.1 & 37123.1
\end{pmatrix}
\begin{pmatrix} u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{Element vectors contribute to}\begin{pmatrix}3\\4\\9\\10\end{pmatrix}\text{ and matrices to}\begin{pmatrix}[3,3]&[3,4]&[3,9]&[3,10]\\[4,3]&[4,4]&[4,9]&[4,10]\\[9,3]&[9,4]&[9,9]&[9,10]\\[10,3]&[10,4]&[10,9]&[10,10]\end{pmatrix}$$

$$\begin{pmatrix}
10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0\\
5.24 & 0 & 117.89 & 0 & 0 & 0 & 0 & 0 & 3.70524 & 0\\
0 & 5.24 & 0 & 117.89 & 0 & 0 & 0 & 0 & 0 & 3.70524\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 3.70524 & 0 & 0 & 0 & 0 & 0 & 7.41048 & 0\\
0 & 0 & 0 & 3.70524 & 0 & 0 & 0 & 0 & 0 & 7.41048
\end{pmatrix}$$

$$\begin{pmatrix}
52500. & 0 & -52500. & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-52500. & 0 & 89623.1 & -37123.1 & 0 & 0 & 0 & 0 & -37123.1 & 37123.1\\
0 & 0 & -37123.1 & 37123.1 & 0 & 0 & 0 & 0 & 37123.1 & -37123.1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -37123.1 & 37123.1 & 0 & 0 & 0 & 0 & 37123.1 & -37123.1\\
0 & 0 & 37123.1 & -37123.1 & 0 & 0 & 0 & 0 & -37123.1 & 37123.1
\end{pmatrix}$$

Element 3

$$\begin{pmatrix}
5.8585 & 0 & 2.92925 & 0\\
0 & 5.8585 & 0 & 2.92925\\
2.92925 & 0 & 5.8585 & 0\\
0 & 2.92925 & 0 & 5.8585
\end{pmatrix}\begin{pmatrix}u_5\\v_5\\u_3\\v_3\end{pmatrix}+\begin{pmatrix}
75131.9 & -37565.9 & -75131.9 & 37565.9\\
-37565.9 & 18783. & 37565.9 & -18783.\\
-75131.9 & 37565.9 & 75131.9 & -37565.9\\
37565.9 & -18783. & -37565.9 & 18783.
\end{pmatrix}\begin{pmatrix}\ddot{u}_5\\\ddot{v}_5\\\ddot{u}_3\\\ddot{v}_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\text{Element vectors contribute to}\begin{pmatrix}9\\10\\5\\6\end{pmatrix}\text{ and matrices to}\begin{pmatrix}[9,9]&[9,10]&[9,5]&[9,6]\\[10,9]&[10,10]&[10,5]&[10,6]\\[5,9]&[5,10]&[5,5]&[5,6]\\[6,9]&[6,10]&[6,5]&[6,6]\end{pmatrix}$$

$$
\begin{pmatrix}
10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10.48 & 0 & 5.24 & 0 & 0 & 0 & 0 & 0 & 0 \\
5.24 & 0 & 117.89 & 0 & 0 & 0 & 0 & 0 & 3.70524 & 0 \\
0 & 5.24 & 0 & 117.89 & 0 & 0 & 0 & 0 & 0 & 3.70524 \\
0 & 0 & 0 & 0 & 5.8585 & 0 & 0 & 0 & 2.92925 & 0 \\
0 & 0 & 0 & 0 & 0 & 5.8585 & 0 & 0 & 0 & 2.92925 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3.70524 & 0 & 2.92925 & 0 & 0 & 0 & 13.269 & 0 \\
0 & 0 & 0 & 3.70524 & 0 & 2.92925 & 0 & 0 & 0 & 13.269
\end{pmatrix}
$$

$$
\begin{pmatrix}
52500. & 0 & -52500. & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-52500. & 0 & 89623.1 & -37123.1 & 0 & 0 & 0 & 0 & -37123.1 & 37123.1 \\
0 & 0 & -37123.1 & 37123.1 & 0 & 0 & 0 & 0 & 37123.1 & -37123.1 \\
0 & 0 & 0 & 0 & 75131.9 & -37565.9 & 0 & 0 & -75131.9 & 37565.9 \\
0 & 0 & 0 & 0 & -37565.9 & 18783. & 0 & 0 & 37565.9 & -18783. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -37123.1 & 37123.1 & -75131.9 & 37565.9 & 0 & 0 & 112255. & -74689. \\
0 & 0 & 37123.1 & -37123.1 & 37565.9 & -18783. & 0 & 0 & -74689. & 55906.1
\end{pmatrix}
$$

Element 4

$$
\begin{pmatrix}
7.86 & 0 & 3.93 & 0 \\
0 & 7.86 & 0 & 3.93 \\
3.93 & 0 & 7.86 & 0 \\
0 & 3.93 & 0 & 7.86
\end{pmatrix}
\begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix}
+
\begin{pmatrix}
0. & 0. & 0. & 0. \\
0. & 70000. & 0. & -70000. \\
0. & 0. & 0. & 0. \\
0. & -70000. & 0. & 70000.
\end{pmatrix}
\begin{pmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{u}_4 \\ \ddot{v}_4 \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Element vectors contribute to $\begin{pmatrix} 3 \\ 4 \\ 7 \\ 8 \end{pmatrix}$ and matrices to $\begin{pmatrix} [3,3] & [3,4] & [3,7] & [3,8] \\ [4,3] & [4,4] & [4,7] & [4,8] \\ [7,3] & [7,4] & [7,7] & [7,8] \\ [8,3] & [8,4] & [8,7] & [8,8] \end{pmatrix}$

Element 5

$$\begin{pmatrix} 7.41048 & 0 & 3.70524 & 0 \\ 0 & 7.41048 & 0 & 3.70524 \\ 3.70524 & 0 & 7.41048 & 0 \\ 0 & 3.70524 & 0 & 7.41048 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_5 \\ v_5 \end{pmatrix} + \begin{pmatrix} 37123.1 & 37123.1 & -37123.1 & -37123.1 \\ 37123.1 & 37123.1 & -37123.1 & -37123.1 \\ -37123.1 & -37123.1 & 37123.1 & 37123.1 \\ -37123.1 & -37123.1 & 37123.1 & 37123.1 \end{pmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_5 \\ \ddot{v}_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element vectors contribute to $\begin{pmatrix} 1 \\ 2 \\ 9 \\ 10 \end{pmatrix}$ and matrices to $\begin{pmatrix} [1,1] & [1,2] & [1,9] & [1,10] \\ [2,1] & [2,2] & [2,9] & [2,10] \\ [9,1] & [9,2] & [9,9] & [9,10] \\ [10,1] & [10,2] & [10,9] & [10,10] \end{pmatrix}$

Element 6

$$\begin{pmatrix} 5.8585 & 0 & 2.92925 & 0 \\ 0 & 5.8585 & 0 & 2.92925 \\ 2.92925 & 0 & 5.8585 & 0 \\ 0 & 2.92925 & 0 & 5.8585 \end{pmatrix} \begin{pmatrix} u_5 \\ v_5 \\ u_4 \\ v_4 \end{pmatrix} + \begin{pmatrix} 75131.9 & 37565.9 & -75131.9 & -37565.9 \\ 37565.9 & 18783. & -37565.9 & -18783. \\ -75131.9 & -37565.9 & 75131.9 & 37565.9 \\ -37565.9 & -18783. & 37565.9 & 18783. \end{pmatrix} \begin{pmatrix} \ddot{u}_5 \\ \ddot{v}_5 \\ \ddot{u}_4 \\ \ddot{v}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Element vectors contribute to $\begin{pmatrix} 9 \\ 10 \\ 7 \\ 8 \end{pmatrix}$ and matrices to $\begin{pmatrix} [9,9] & [9,10] & [9,7] & [9,8] \\ [10,9] & [10,10] & [10,7] & [10,8] \\ [7,9] & [7,10] & [7,7] & [7,8] \\ [8,9] & [8,10] & [8,7] & [8,8] \end{pmatrix}$

Boundary conditions

Node	dof	Value
1	u_1	0
	v_1	0
3	u_3	0
	v_3	0
4	u_4	0
	v_4	0

Remove {1, 2, 5, 6, 7, 8} rows and columns.

Equations of motion

$$\begin{pmatrix} 125.75 & 0 & 3.70524 & 0 \\ 0 & 125.75 & 0 & 3.70524 \\ 3.70524 & 0 & 26.538 & 0 \\ 0 & 3.70524 & 0 & 26.538 \end{pmatrix} \begin{pmatrix} \ddot{u}_2 \\ \ddot{v}_2 \\ \ddot{u}_5 \\ \ddot{v}_5 \end{pmatrix} + \begin{pmatrix} 89623.1 & -37123.1 & -37123.1 & 37123.1 \\ -37123.1 & 107123. & 37123.1 & -37123.1 \\ -37123.1 & 37123.1 & 224510. & 0 \\ 37123.1 & -37123.1 & 0 & 111812. \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ u_5 \\ v_5 \end{pmatrix} = \begin{pmatrix} 10000. \\ 17320.5 \\ 0 \\ 0 \end{pmatrix}$$

2. Space truss element

Local element coordinates   **Global coordinates**
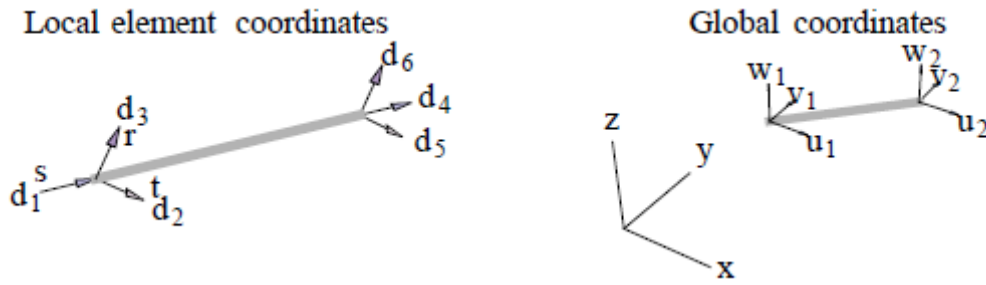
$$
m_\ell = \begin{pmatrix}
\frac{Lm}{3} & 0 & 0 & \frac{Lm}{6} & 0 & 0 \\
0 & \frac{Lm}{3} & 0 & 0 & \frac{Lm}{6} & 0 \\
0 & 0 & \frac{Lm}{3} & 0 & 0 & \frac{Lm}{6} \\
\frac{Lm}{6} & 0 & 0 & \frac{Lm}{3} & 0 & 0 \\
0 & \frac{Lm}{6} & 0 & 0 & \frac{Lm}{3} & 0 \\
0 & 0 & \frac{Lm}{6} & 0 & 0 & \frac{Lm}{3}
\end{pmatrix}
$$

$$
k_\ell = \begin{pmatrix}
\frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{pmatrix} = \begin{pmatrix} \ell_s & m_s & n_s & 0 & 0 & 0 \\ \ell_t & m_t & n_t & 0 & 0 & 0 \\ \ell_r & m_r & n_r & 0 & 0 & 0 \\ 0 & 0 & 0 & \ell_s & m_s & n_s \\ 0 & 0 & 0 & \ell_t & m_t & n_t \\ 0 & 0 & 0 & \ell_r & m_r & n_r \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{d_\ell = T\, d}$$

$$m\,\ddot{d} + k\,d = 0$$

$$\mathbf{k} = \boldsymbol{T^T k_\ell T} = \frac{EA}{L} \begin{pmatrix} \ell_s^2 & m_s\ell_s & n_s\ell_s & -\ell_s^2 & -m_s\ell_s & -n_s\ell_s \\ m_s\ell_s & m_s^2 & m_s n_s & -m_s\ell_s & -m_s^2 & -m_s n_s \\ n_s\ell_s & m_s n_s & n_s^2 & -n_s\ell_s & -m_s n_s & -n_s^2 \\ -\ell_s^2 & -m_s\ell_s & -n_s\ell_s & \ell_s^2 & m_s\ell_s & n_s\ell_s \\ -m_s\ell_s & -m_s^2 & -m_s n_s & m_s\ell_s & m_s^2 & m_s n_s \\ -n_s\ell_s & -m_s n_s & -n_s^2 & n_s\ell_s & m_s n_s & n_s^2 \end{pmatrix}$$