Lecture 22: Finite element method: Axial vibrations of bars

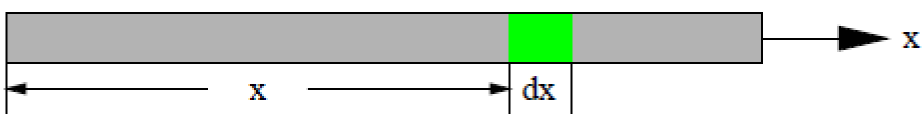
Reading materials: Section 9.1

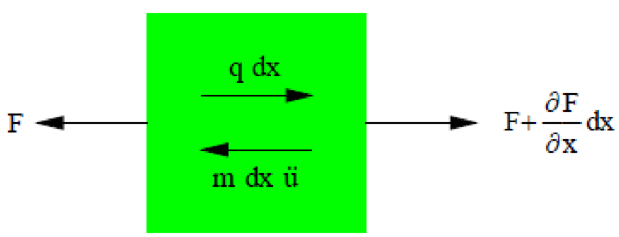
1. Introduction

- Discretization
- Assembly and solution

2. Governing equations

- Axial vibrations of a long slender bar





$$\frac{\partial}{\partial x}\left(A\,E\,\frac{\partial u}{\partial x}\right) + q = m\;\ddot{u}$$

- initial conditions

$$u(x,\,0) = u_0 \qquad \dot{u}(x,\,0) = v_0$$

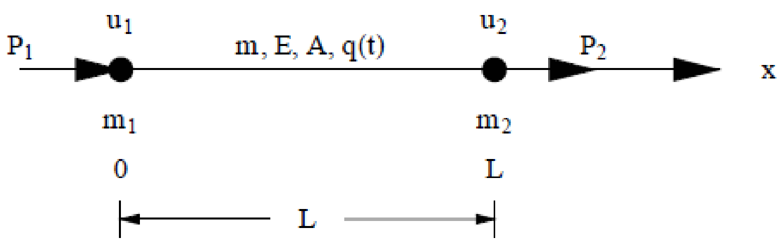
boundary conditions

$$u(x_\ell, t) = u_{x_\ell} \quad \text{or} \quad A(x_\ell)\,E(x_\ell)\,\frac{\partial u(x_\ell,t)}{\partial x} = F_{x\ell}(t); \quad x_\ell \rightarrow \text{right end of the bar}$$

$$-A(x_0)\,E(x_0)\,\frac{\partial u(x_0,t)}{\partial x} = F_{x_0}(t); \quad x_0 \rightarrow \text{left end of the bar}$$

$u(x_0, t)=u_{x0}$

3. Axial vibration element



Finite element approximation

$$y(x,\, t) = \sum_{i=1}^{2} N_i(x)\, u_i(t) \equiv \boldsymbol{N}^T \boldsymbol{u}$$

$$\text{y(x)} = a_0 + x\, a_1$$
$$\text{y(0)} = u_1 \Longrightarrow a_0 = u_1$$
$$\text{y(L)} = u_2 \Longrightarrow a_0 + L\, a_1 = u_2$$

$$\left\{a_1 \rightarrow -\frac{u_1 - u_2}{L},\ a_0 \rightarrow u_1\right\}$$

$$\text{y(x)} = u_1 - \frac{x\,(u_1 - u_2)}{L} = \left(1 - \frac{x}{L}\right) u_1 + \frac{x\, u_2}{L}$$

$$N^{\mathrm{T}} = \left\{1 - \frac{x}{L},\ \frac{x}{L}\right\}$$

$$\frac{\mathrm{dy}(x,t)}{\mathrm{dx}} = \sum_{i=1}^{2} \frac{\mathrm{dN}_i}{\mathrm{dx}}\, u_i(t) \equiv \boldsymbol{B}^T \boldsymbol{u}$$

$$\boldsymbol{B}^{\mathrm{T}} = \left\{-\frac{1}{L}, \frac{1}{L}\right\}$$

4. Energy

- kinetic energy due to distributed mass

$$T_d = \frac{1}{2}\int_0^L m\left(\frac{\mathrm{dy}(x,t)}{\mathrm{dt}}\right)^2 \mathrm{dx} = \frac{1}{2}\,\dot{\boldsymbol{u}}^T \int_0^L m\,\boldsymbol{N}\,\boldsymbol{N}^T\,\mathrm{dx}\,\dot{\boldsymbol{u}} \equiv \frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}_d\,\dot{\boldsymbol{u}}$$

$$\boldsymbol{m}_d = \int_0^L \mathrm{m}\,\mathrm{N}\,\boldsymbol{N}^T\,\mathrm{dx} = \int_0^L \begin{pmatrix} m\left(1-\frac{x}{L}\right)^2 & \frac{m\,x\left(1-\frac{x}{L}\right)}{L} \\ \frac{m\,x\left(1-\frac{x}{L}\right)}{L} & \frac{m\,x^2}{L^2} \end{pmatrix} \mathrm{dx} = \begin{pmatrix} \frac{L\,m}{3} & \frac{L\,m}{6} \\ \frac{L\,m}{6} & \frac{L\,m}{3} \end{pmatrix}$$

- kinetic energy due to concentrated mass

$$T_c = \frac{1}{2}\left(m_1\,u_1^2 + m_2\,u_2^2\right) = \frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}_c\,\dot{\boldsymbol{u}} \qquad \boldsymbol{m}_c = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$$

- Total kinetic energy

$$T = T_d + T_c = \frac{1}{2}\,\dot{\boldsymbol{u}}^T(\boldsymbol{m}_d + \boldsymbol{m}_c)\,\dot{\boldsymbol{u}} \equiv \frac{1}{2}\,\dot{\boldsymbol{u}}^T\,\boldsymbol{m}\,\dot{\boldsymbol{u}}$$

$$\boldsymbol{m} = \boldsymbol{m}_d + \boldsymbol{m}_c = \begin{pmatrix} \frac{L\,m}{3} + m_1 & \frac{L\,m}{6} \\ \frac{L\,m}{6} & \frac{L\,m}{3} + m_2 \end{pmatrix}$$

- Strain energy

$$U_s = \frac{1}{2}\int_0^L EA\left(\frac{dy(x,t)}{dx}\right)^2 dx = \frac{1}{2}\boldsymbol{u}^T\int_0^L EA\,\boldsymbol{B}\,\boldsymbol{B}^T dx\,\boldsymbol{u} \equiv \frac{1}{2}\boldsymbol{u}^T\boldsymbol{k}\,\boldsymbol{u}$$

$$\mathbf{k} = \int_0^L EA\,\mathbf{B}\,\boldsymbol{B}^T dx = \int_0^L \begin{pmatrix} \frac{AE}{L^2} & -\frac{AE}{L^2} \\ -\frac{AE}{L^2} & \frac{AE}{L^2} \end{pmatrix} dx = \frac{AE}{L}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

- work done by the distributed applied force

$$W_q = \int_0^L q(t)\,y(x,t)\,dx = \int_0^L q\,\boldsymbol{N}^T dx\,\boldsymbol{u} \equiv \boldsymbol{r}_q^T\boldsymbol{u}$$

$$\boldsymbol{r}_q^T = \int_0^L q\,N^T dx = \int_0^L \left\{q\left(1-\frac{x}{L}\right), \frac{q\,x}{L}\right\} dx = \left\{\frac{L\,q}{2}, \frac{L\,q}{2}\right\}$$

- work done by the concentrated loads

$$W_p = (p_1 u_1 + p_2 u_2) = \boldsymbol{r}_p^T\boldsymbol{u}$$

$$\boldsymbol{r}_p^T = (\,p_1 \quad p_2\,)^T$$

- Total work done

$$W = W_q + W_p = (\boldsymbol{r}_q + \boldsymbol{r}_p)^T\boldsymbol{u} \equiv \boldsymbol{r}^T\boldsymbol{u}$$

$$\boldsymbol{r} = \boldsymbol{r}_q + \boldsymbol{r}_p = \begin{pmatrix} \frac{L\,q}{2} + p_1 \\ \frac{L\,q}{2} + p_2 \end{pmatrix}$$

5. Equations of motion

$U=U_s-W$

$$\frac{d}{dt}(T+U)=0 \quad \Longrightarrow \quad \frac{d}{dt}\left[\frac{1}{2}\dot{\boldsymbol{u}}^T \boldsymbol{m}\,\dot{\boldsymbol{u}}+\frac{1}{2}\boldsymbol{u}^T \boldsymbol{k}\,\boldsymbol{u}-\boldsymbol{r}^T \boldsymbol{u}\right]=\dot{\boldsymbol{u}}^T[\boldsymbol{m}\,\ddot{\boldsymbol{u}}+\boldsymbol{k}\,\boldsymbol{u}-\boldsymbol{r}]=\boldsymbol{0}$$

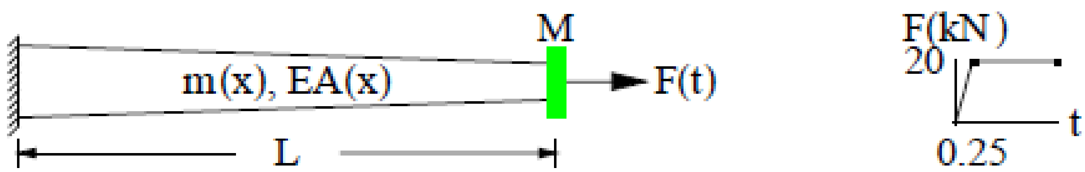
$$\boldsymbol{m}\,\ddot{\boldsymbol{u}}(t)+\boldsymbol{k}\,\boldsymbol{u}(t)=\boldsymbol{r}(t)$$

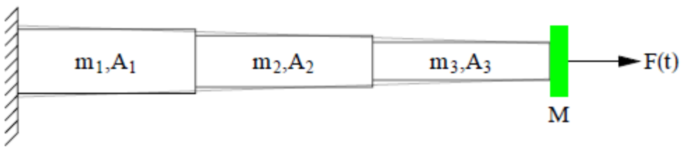
$$\begin{pmatrix} \frac{L\,m}{3}+m_1 & \frac{L\,m}{6} \\ \frac{L\,m}{6} & \frac{L\,m}{3}+m_2 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix}+\begin{pmatrix} \frac{\mathrm{EA}}{L} & -\frac{\mathrm{EA}}{L} \\ -\frac{\mathrm{EA}}{L} & \frac{\mathrm{EA}}{L} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} \frac{L\,q}{2}+p_1 \\ \frac{L\,q}{2}+p_2 \end{pmatrix}$$

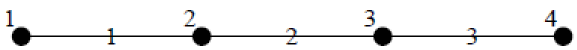
6. Assembly and Solution

- The element equations are derived in the above with local coordinates.
- Assemble the element equations into the global equations based on the global coordinates.
- Apply the boundary conditions.
- Numerically solve the equations of motion.

7. Example 1







- General element equations

$$\begin{pmatrix} \frac{Lm}{3} & \frac{Lm}{6} \\ \frac{Lm}{6} & \frac{Lm}{3} \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{Lq}{2} \\ \frac{Lq}{2} \end{pmatrix}$$

- Element 1

$$\begin{pmatrix} 1.869 & 0.9345 \\ 0.9345 & 1.869 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{pmatrix} 490000 & -490000 \\ -490000 & 490000 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Element 2

$$\begin{pmatrix} 1.335 & 0.6675 \\ 0.6675 & 1.335 \end{pmatrix}\begin{pmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{pmatrix} + \begin{pmatrix} 350000 & -350000 \\ -350000 & 350000 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Element 3

$$\begin{pmatrix} 0.801 & 0.4005 \\ 0.4005 & 0.801 \end{pmatrix}\begin{pmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 210000 & -210000 \\ -210000 & 210000 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Considering concentrated mass and force on node 4, the element equations for Element 3 could be written as:

$$\begin{pmatrix} 0.801 & 0.4005 \\ 0.4005 & 100.801 \end{pmatrix}\begin{pmatrix} \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 210000 & -210000 \\ -210000 & 210000 \end{pmatrix}\begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ F(t) \end{pmatrix}$$

Form of global equations

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Assemble of elements

Element 1

$$\text{lm} = (1 \quad 2)$$

$$\text{Assembly locations for } \boldsymbol{k} \;\&\; \boldsymbol{m} \Longrightarrow \begin{pmatrix} [1, 1] & [1, 2] \\ [2, 1] & [2, 2] \end{pmatrix} \text{ and for } \boldsymbol{r} \Longrightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 490000. & -490000. & 0 & 0 \\ -490000. & 490000. & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Element 2

lm = (2 3)

Assembly locations for $\boldsymbol{k}$ & $\boldsymbol{m} \Longrightarrow \begin{pmatrix} [2,2] & [2,3] \\ [3,2] & [3,3] \end{pmatrix}$ and for $\boldsymbol{r} \Longrightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$$\boldsymbol{k} = \begin{pmatrix} 490000. & -490000. & 0 & 0 \\ -490000. & 840000. & -350000. & 0 \\ 0 & -350000. & 350000. & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Element 3

lm = (3 4)

Assembly locations for $\boldsymbol{k}$ & $\boldsymbol{m} \Longrightarrow \begin{pmatrix} [3,3] & [3,4] \\ [4,3] & [4,4] \end{pmatrix}$ and for $\boldsymbol{r} \Longrightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$$\boldsymbol{m} = \begin{pmatrix} 1.869 & 0.9345 & 0 & 0 \\ 0.9345 & 3.204 & 0.6675 & 0 \\ 0 & 0.6675 & 2.136 & 0.4005 \\ 0 & 0 & 0.4005 & 100.801 \end{pmatrix}$$

$$\boldsymbol{k} = \begin{pmatrix} 490000. & -490000. & 0 & 0 \\ -490000. & 840000. & -350000. & 0 \\ 0 & -350000. & 560000. & -210000. \\ 0 & 0 & -210000. & 210000. \end{pmatrix}$$

Global equations of motion

$$\begin{pmatrix} 1.869 & 0.9345 & 0 & 0 \\ 0.9345 & 3.204 & 0.6675 & 0 \\ 0 & 0.6675 & 2.136 & 0.4005 \\ 0 & 0 & 0.4005 & 100.801 \end{pmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 490\,000. & -490\,000. & 0 & 0 \\ -490\,000. & 840\,000. & -350\,000. & 0 \\ 0 & -350\,000. & 560\,000. & -210\,000. \\ 0 & 0 & -210\,000. & 210\,000. \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ F(t) \end{pmatrix}$$

Apply BCs

$$u_1 = \ddot{u}_1 = 0$$

$$\begin{pmatrix} 3.204 & 0.6675 & 0 \\ 0.6675 & 2.136 & 0.4005 \\ 0 & 0.4005 & 100.801 \end{pmatrix} \begin{pmatrix} \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 840\,000. & -350\,000. & 0 \\ -350\,000. & 560\,000. & -210\,000. \\ 0 & -210\,000. & 210\,000. \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F(t) \end{pmatrix}$$

Note: for nonzero BCs

$$\begin{pmatrix} 3.204 & 0.6675 & 0 \\ 0.6675 & 2.136 & 0.4005 \\ 0 & 0.4005 & 100.801 \end{pmatrix} \begin{pmatrix} \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{pmatrix} + \begin{pmatrix} 840\,000. & -350\,000. & 0 \\ -350\,000. & 560\,000. & -210\,000. \\ 0 & -210\,000. & 210\,000. \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ u_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ F(t) \end{pmatrix} - \begin{pmatrix} 0.9345 \\ 0 \\ 0 \end{pmatrix} \ddot{u}_1 - \begin{pmatrix} -490\,000. \\ 0 \\ 0 \end{pmatrix} u_1$$

Solutions

Newmark's method