

Lecture 20: Energy Method

Reading materials: Sections 8.1, 8.2, and 8.3

1. Rayleigh's energy method

- Rayleigh's method is based on the principle of conservation of energy.
- The energy in a dynamic system consists of the kinetic energy and the potential energy.

$$T + U = \text{constant}$$

- The kinetic energy is stored in the mass and is proportional to the square of the velocity. The potential energy includes strain energy that is proportional to elastic deformations and the work done by the applied forces.

- The total energy must remain constant.

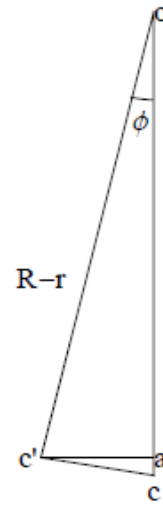
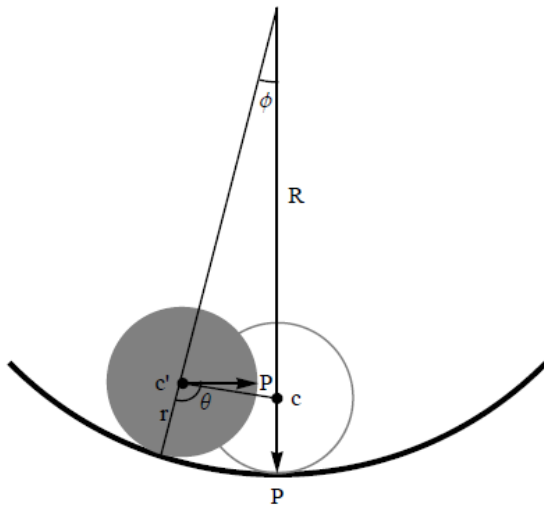
$$T_1 + U_1 = T_2 + U_2$$

$$\frac{d}{dt} (T + U) = 0$$

- For a simple spring-mass system, the potential energy is zero when the mass is passing through the static equilibrium position. At this time, the acceleration is zero while the velocity is maximum. Therefore, the kinetic energy is at its maximum. When the mass is at its extreme position, the kinetic energy is zero and the potential energy is at its maximum.

$$T_{max} = U_{max}$$

● Example 1:



2. Rayleigh's energy method: Effective mass

● The energy method can be used for multimass systems or for distributed mass systems, provided the motion of every point in the system is known.

● In systems in which masses are joined by rigid links, levers, or gears, the motion of various masses can be expressed in terms of the motion \dot{x} of some specific point and the system is simply one of a single DOF, because only one coordinate is necessary. The kinetic energy can then be written as

$$T = \frac{1}{2} m_{eff} \dot{x}^2$$

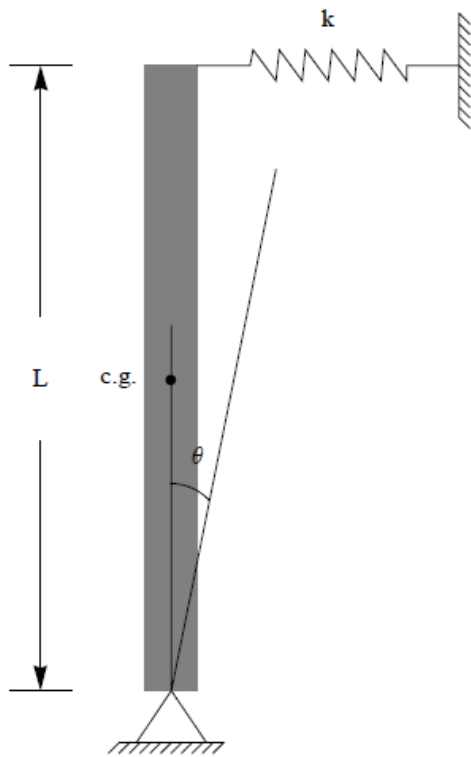
where m_{eff} is the effective mass or an equivalent lumped mass at the specified point.

● If the stiffness at that point is also known, the natural frequency can be calculated from the simple equation

$$\omega_n = \sqrt{\frac{k}{m_{eff}}}$$

● Example: Determine the effect of the mass of the spring on the natural frequency.

Example:



3. Principle of virtual work (Optional)

● The principle of virtual work is associated with the equilibrium of bodies, and may be stated as follows: If a system in equilibrium under the action of a set of forces is given a virtual displacement, the virtual work done by the forces will be zero.

● The terms used in this statement are defined as follows.

● A virtual displacement δr is an imaginary infinitesimal variation of the coordinate given instantaneously. The virtual displacement must be compatible with the constraints of the system.

● Virtual work δW is the work done by all the active forces in a virtual displacement. Because there is no significant change of geometry associated with the virtual displacement, the forces acting on the system are assumed to remain unchanged for the calculation of δW .

● Example

4. Rayleigh's energy method for continuous systems

• A continuous system is approximated by an equivalent single degree of freedom. The assumption is regarding deformation shape of this system

• beam bending, the deflection is assumed to be

$$y(x, t) = \psi(x) u(t)$$

Strain energy

$$U = \frac{1}{2} \int_0^L EI \left[\frac{d^2 y(x,t)}{dx^2} \right]^2 dx = \frac{1}{2} [u(t)]^2 \int_0^L EI \left[\frac{d^2 \psi}{dx^2} \right]^2 dx \equiv \frac{1}{2} \bar{k} [u(t)]^2$$

where

$$\bar{k} = \int_0^L EI \left[\frac{d^2 \psi}{dx^2} \right]^2 dx$$

Kinetic energy

$$T = \frac{1}{2} \int_0^L m \left[\frac{dy(x,t)}{dt} \right]^2 dx = \frac{1}{2} \left(\frac{du}{dt} \right)^2 \int_0^L m [\psi(x)]^2 dx \equiv \frac{1}{2} \bar{m} \left(\frac{du}{dt} \right)^2$$

where

$$\bar{m} = \int_0^L m [\psi(x)]^2 dx$$

Equations of motion

$$\frac{d}{dt} (T + U) = 0 \implies \frac{d}{dt} \left[\frac{1}{2} \bar{m} \left(\frac{du}{dt} \right)^2 + \frac{1}{2} \bar{k} [u(t)]^2 \right] = 0$$

$$\bar{m} \ddot{u}(t) + \bar{k} u(t) = 0$$

$$\omega = \sqrt{\frac{\bar{k}}{\bar{m}}} = \sqrt{\frac{\int_0^L EI \left[\frac{d^2 \psi}{dx^2} \right]^2 dx}{\int_0^L m [\psi(x)]^2 dx}}$$

🟢 Example of fixed-pinned beam

5. Alternate form of the Rayleigh's energy method

• For a conservative system, the total energy remains constant.

$$T + U = \text{constant}$$

• At different time, for example, time 1 and 2, we have


$$T_1 + U_1 = T_2 + U_2$$

• For a simple spring-mass system, we have

$$T_{\max} = U_{\max}$$

• Rayleigh's method for continuous systems needs integration to compute equivalent mass and stiffness. To avoid integration in the energy method for continuous systems, the mass is assumed to be lumped at few points. The static deflected shape is computed by applying concentrated loads on those points.

• Beam bending

 Example