Lecture 17: Response Spectra

Reading materials: Sections 6.1, 6.2, and 6.3

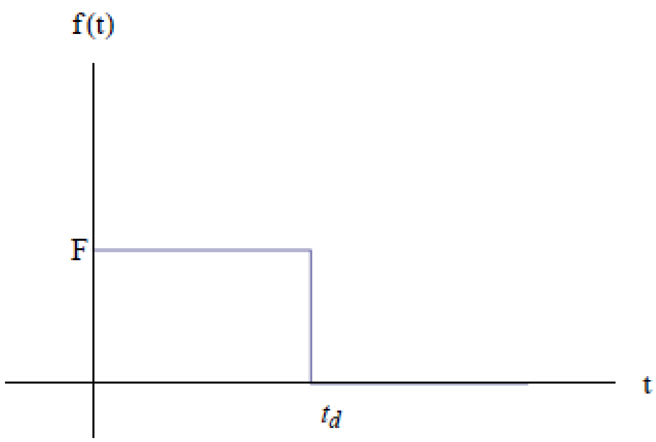
1. Concepts

- In practical dynamic analysis situations we are interested in the maximum response.

- The graph showing the variation of the maximum response (maximum displacement, velocity, acceleration, or any other quantity) with the natural frequency (or natural period) of a single degree of freedom system to a specified forcing function is known as the response spectrum.

- A response spectrum is a plot of maximum response of a single degree of freedom system subject to a specific input, such as step loading and triangular pulse versus period of vibration or another suitable quantity.

- Example: Response spectra for a rectangular pulse loading

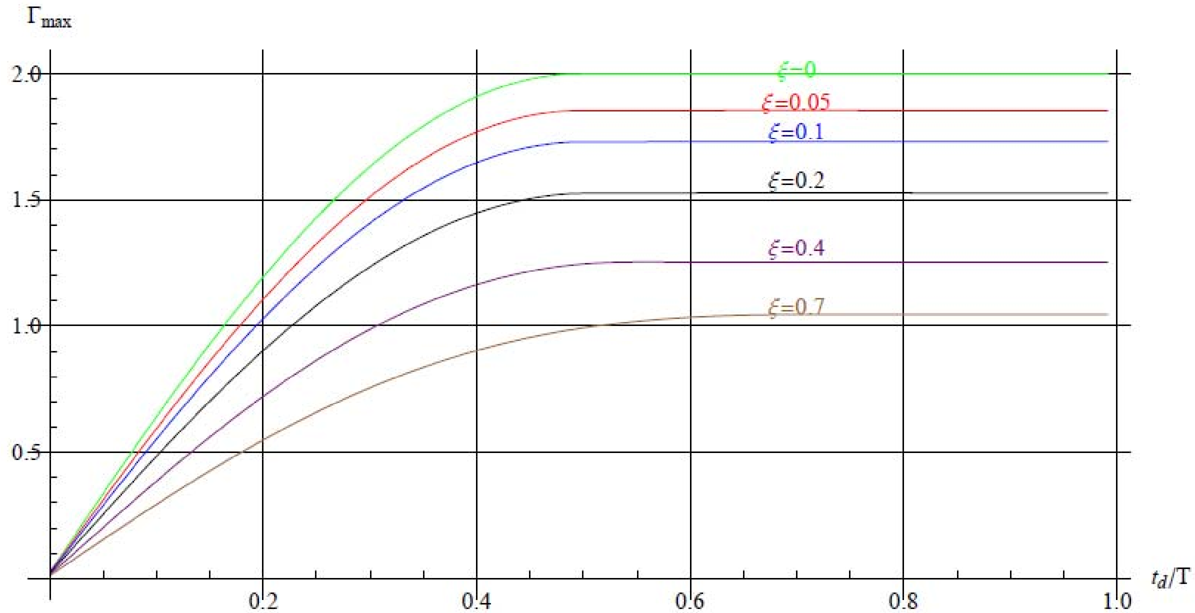


T: fundamental period of the structure

u_{max}: maxium deflection over time

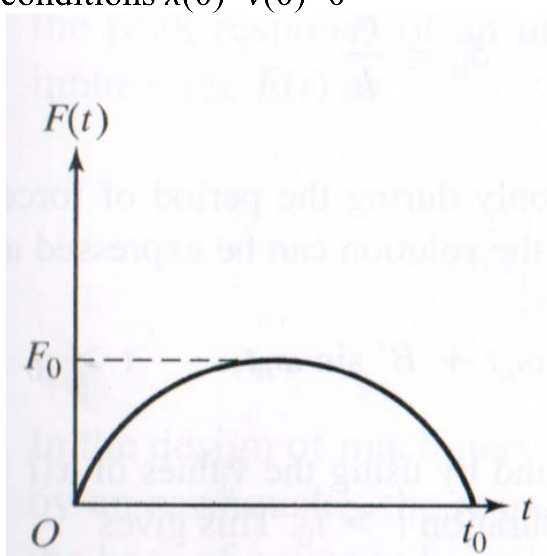
u_{static}: deflection if load F is treated as a static load

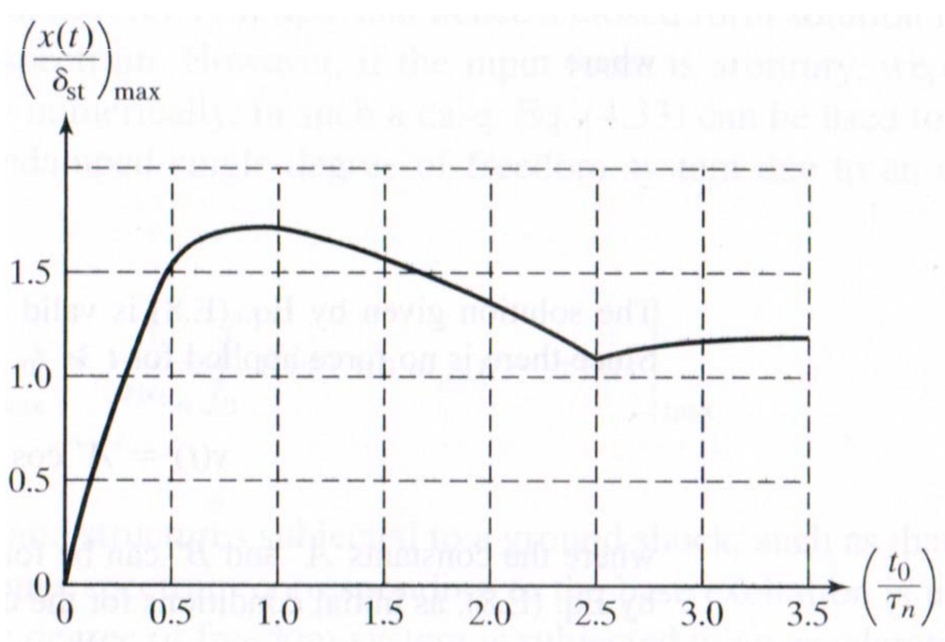
Γ_{max}: maximum dynamic load magnification factor



2. Response Spectrum of Sinusoidal Pulse

Find the response spectrum for the sinusoidal pulse force using the initial conditions $x(0)=v(0)=0$





(b)

3. Usage of Response Spectrum

- SDOF systems

Period of vibration:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$$

Maximum displacement

$$\Gamma_{\max} \equiv \frac{u_{\max}}{u_{\text{static}}} \implies u_{\max} = \Gamma_{\max} u_{\text{static}}$$

- MDOF systems

Equations of motion

$$\boldsymbol{m}\,\ddot{\boldsymbol{y}}(t) + (\alpha\,\boldsymbol{m} + \beta\,\boldsymbol{k})\,\dot{\boldsymbol{y}}(t) + \boldsymbol{k}\,\boldsymbol{y}(t) = \boldsymbol{f}\,(t);$$

Undamped free vibration mode shapes and frequencies

$$\boldsymbol{k}\,\boldsymbol{\phi}_i = \lambda_i\,\boldsymbol{m}\,\boldsymbol{\phi}_i; \quad i = 1, 2, \ldots, n$$

Modal coordinates

$$\boldsymbol{z} = (\,z_1 \quad z_2 \quad \ldots \quad z_n\,)^T$$

$$\boldsymbol{y}\,(t) = \textstyle\sum_i z_i(t)\,\boldsymbol{\phi}_i$$

Damped modal equations

$$M_i \ddot{z}_i(t) + (\alpha M_i + \beta K_i)\dot{z}_i(t) + K_i z_i(t) = F_i(t); \quad i = 1, 2, \ldots$$

$$M_i = \boldsymbol{\phi}_i^T \boldsymbol{m}\, \boldsymbol{\phi}_i; \quad K_i = \boldsymbol{\phi}_i^T \boldsymbol{k}\, \boldsymbol{\phi}_i; \quad \omega_i = \sqrt{K_i / M_i}\,; \quad F_i = \boldsymbol{\phi}_i^T \boldsymbol{f}$$

Solution

$$\boldsymbol{u}(t) = \sum_i z_i(t)\, \boldsymbol{\phi}_i$$

We know

$$z_{i,\max}$$

Here,

$$\boldsymbol{u}_{\max} \neq \sum_i z_{i,\max}\, \boldsymbol{\phi}_i$$

$$u_{n,\max} = \sqrt{\sum_i \left(z_{i,\max} (\phi_i)_n\right)^2}\,; \quad n = 1, 2, \ldots$$

4. Response spectra using Duhamel's integral

- In the above examples, the input force is simple and hence a closed form solution has been obtained for the response spectrum. If the input force is arbitrary, we can find the response spectrum only numerically.

- The peak displacement response of an undamped SDOF system subjected to a given load F(t) can be expressed via Duhamel's integral

- Loading phase: $t < t_d$

$$|\, y(t)\,|_{\max} = \left|\, \frac{1}{m\,\omega} \int_0^t F(\tau) \sin \omega (t - \tau)\, \mathrm{d}\tau \,\right|_{\max}$$

$$|\Gamma|_{max} = \left|\frac{y(t)}{F_0/k}\right|_{max} = \left|\frac{k}{F_0}\frac{1}{m\omega}\int_0^t F(t)\sin\omega(t-T)\,dT\right|_{max} = \left|\frac{\omega}{F_0}\int_0^t F(t)\sin\omega(t-T)\,dT\right|_{max}$$

where F_0 is the load magnitude.

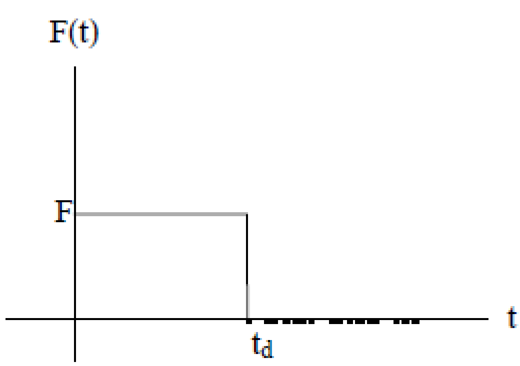
- Free vibration phase: $t >= t_d$

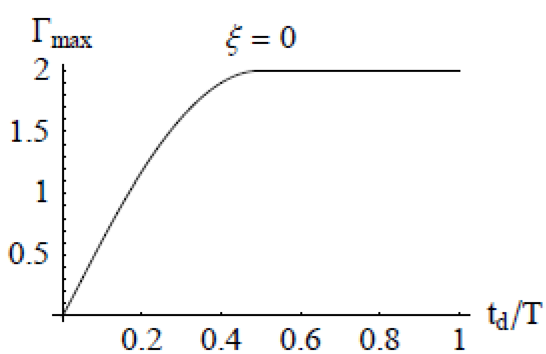
$$|y(t)|_{max} = \left|u_{t_d}\cos\omega(t-t_d) + \frac{v_{t_d}}{\omega}\sin\omega(t-t_d)\right|_{max}$$

$$|\Gamma|_{max} = \left|\frac{y(t)}{F_0/k}\right|_{max} = \left|\frac{m\omega^2}{F_0}\left(u_{t_d}\cos\omega(t-t_d) + \frac{v_{t_d}}{\omega}\sin\omega(t-t_d)\right)\right|_{max}$$

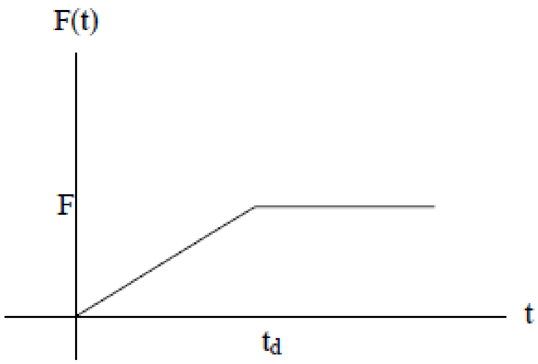
where u_{td} and v_{td} are displacement and velocity at the end of the forced vibration phase.

- Rectangular Pulse




Γmax
ξ = 0
2
1.5
1
0.5
0.2
0.4
0.6
0.8
1
td/T

Step force with ramp: maximum DLF occurs at the constant load phase.



- **Ramp loading phase: $t < t_d$**

$$y(t) = \frac{1}{m\omega}\int_0^t F(\tau/t_d)\sin(\omega(t-\tau))d\tau = \frac{F\,(t\,\omega - \sin(t\,\omega))}{m\,\omega^3\,t_d}$$

$$y(t) = \frac{F\,(t\,\omega - \sin(t\,\omega))}{k\,\omega\,t_d}$$

$$\Gamma(t) = \frac{y(t)}{F/k} = \frac{t\,\omega - \sin(t\,\omega)}{\omega\,t_d}$$

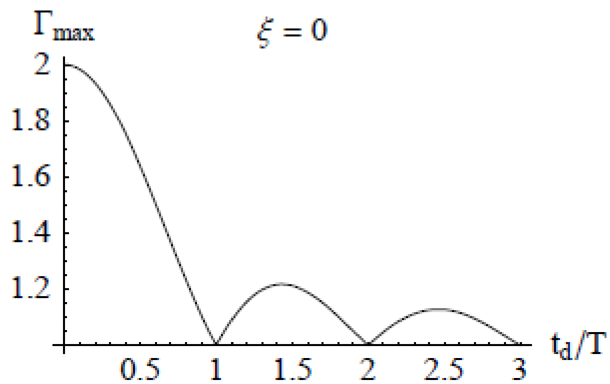
- **Constant loading phase: $t \geq t_d$**

$$y(t) = u_{t_d}\cos(\omega(t-t_d)) + \frac{v_{t_d}}{\omega}\sin(\omega(t-t_d)) + \frac{F}{m\omega}\int_{t_d}^t \sin(\omega(t-\tau))d\tau$$

$$y(t_d) = u_{t_d} = \frac{F\left(\omega\,t_d - \sin(\omega\,t_d)\right)}{k\,\omega\,t_d}; \qquad \dot{y}(t_d) = v_{t_d} = \frac{F\left(\omega - \omega\cos(\omega\,t_d)\right)}{k\,\omega\,t_d}$$

$$\Gamma(t) = \frac{y(t)}{F/k} = -\frac{\sin(t\,\omega)}{\omega\,t_d} + \frac{\sin\left(\omega\left(t - t_d\right)\right)}{\omega\,t_d} + 1$$

$$\Gamma_{max} = \frac{1}{2}\left|\frac{2\sin(\omega(t-t_d)) + 2\omega t_d - \sqrt{2-2\cos(\omega t_d)}}{\omega t_d}\right| \equiv \frac{\sqrt{1-\cos(2\,t_d/T\,\pi)}}{\sqrt{2}\;t_d/T\,\pi} + 1$$



5. Response spectra using Numerical Integration

- It is difficult to determine simple analytical expressions for maximum DLF for complicated loading.

- Equation of motion for a SDOF system

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = f(t); \quad u(0) = 0; \quad \dot{u}(0) = 0$$

$$\ddot{u}(t) + 2\,\xi\,\omega\,\dot{u}(t) + \omega^2\,u(t) = \frac{f(t)}{m}$$

- Let

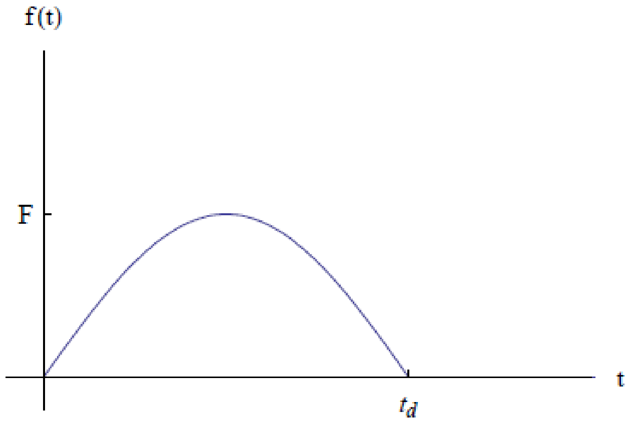
$$\frac{t_d}{T} = t_d; \quad \omega = \frac{2\pi}{T} = 2\pi; \quad \omega^2 = 4\pi^2$$

$$\frac{F}{k} = 1 \quad \Longrightarrow \quad F = k = m\,\omega^2 = 4\pi^2$$

then

$$\ddot{u}(t) + 2\,\xi\,(2\pi)\,\dot{u}(t) + 4\pi^2\,u(t) = f(t)$$

Half Sine Pulse

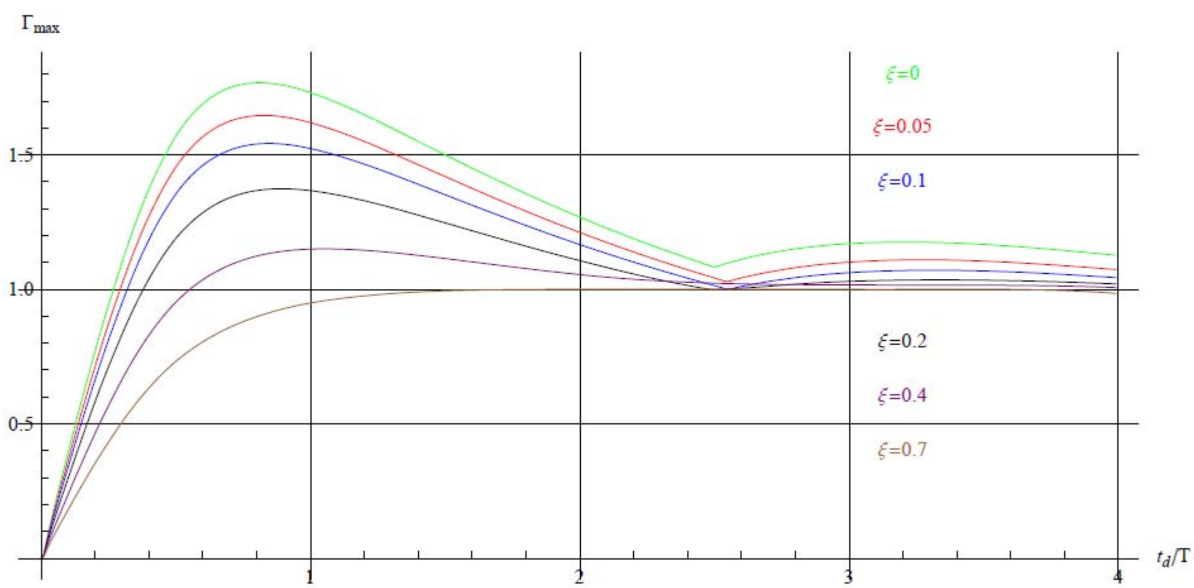


Equation of motion

$$\ddot{u}(t) + 2\,\xi\,(2\,\pi)\,\dot{u}(t) + 4\,\pi^2\,u(t) = f\,(t)$$

$$f(t) = 4\,\pi^2 \sin(\pi\, t\,/\,t_d); \quad 0 \le t \le t_d$$

$$f(t) = 0 \quad t > t_d$$



$t_d = 0.25$ and $\xi = 0.1$ the solution is as follows.

$k = 4\pi^2$; $m = 1$; $\omega = 2\pi$ rad/s; $T = 1$ s

$\beta = 0.25$; $\gamma = 0.5$; $\Delta t = 0.1$

t	F	Disp	Vel	Acc
0	0.	0	0	0.
0.025	12.1995	0.00186536	0.149229	11.9383
0.05	23.2048	0.0109072	0.574118	22.0528
0.075	31.9387	0.0332529	1.21354	29.101
0.1	37.5462	0.0731651	1.97943	32.1703
0.125	39.4784	0.132486	2.76621	30.772
0.15	37.5462	0.210338	3.46201	24.8919
0.175	31.9387	0.303121	3.96059	14.995
0.2	23.2048	0.404788	4.17279	1.98075
0.225	12.1995	0.507401	4.03625	−12.904
0.25	4.83455×10^{-15}	0.601887	3.5226	−28.1882
0.275	0.	0.6808	2.79045	−30.3835
0.3	0.	0.740848	2.01344	−31.7777
0.325	0.	0.781163	1.21169	−32.3617
0.35	0.	0.801376	0.405343	−32.1464
0.375	0.	0.801617	−0.386006	−31.1615
0.4	0.	0.782496	−1.14371	−29.4545
0.425	0.	0.745068	−1.8505	−27.0887
0.45	0.	0.690801	−2.49087	−24.1416
0.475	0.	0.621523	−3.05142	−20.7022
0.5	0.	0.539367	−3.52106	−16.8686