Lecture 16: Numerical Solution

Reading materials: Section 5.3

1. Introduction

- For complex loading time histories, the closed-form solutions become impossible to obtain and therefore we must resort to numerical methods.

- All numerical methods compute solution at discrete time steps and are based on some assumption regarding the solution over a given time interval.

- The choice of a suitable time step is critical.

- It is important to understand Accuracy and Stability of numerical methods.

- An accurate numerical solution is close to the exact solution of the differential equation.

- The stability refers to the largest time step that can be used without solution becoming unbounded due to accumulation of errors.

- An unconditionally stable method results in the solution staying bounded even with very large time step.

- For conditionally stable methods, the stability criteria are generally defined in terms of natural frequencies or period of vibration.

- Equations of motion

$$m\ddot{u} + c\dot{u} + ku = f(t); \quad u(0) = u_0; \quad \dot{u}(0) = v_0$$

- The solution at time t_i is known: $(u_i, \dot{u}_i, \ddot{u}_i)$

- The solution at time t_{i+1} is unknown: $(u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1})$

2. Newmark's constant average acceleration method

- The acceleration is assumed to be constant over the interval time.

$$m\,\ddot{u} + c\,\dot{u} + k\,u = f(t);\quad u(0) = u_0;\quad \dot{u}(0) = v_0$$

- Numerically updates from t_i to t_{i+1}

At time t_i, the acceleration, velocity and displacement are known. The force is prescribed.

$$\left(m + \frac{\Delta t}{2}\,c + \frac{\Delta t^2}{4}\,k\right)\ddot{u}_{i+1} = f_{i+1} - c\left(\dot{u}_i + \frac{\Delta t}{2}\,\ddot{u}_i\right) - k\left(u_i + \Delta t\,\dot{u}_i + \frac{\Delta t^2}{4}\,\ddot{u}_i\right)$$

$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)$$

$$u_{i+1} = u_i + \Delta t\,\dot{u}_i + \frac{\Delta t^2}{4}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)$$

For Multiple degree of freedom systems

$$\left(\boldsymbol{m} + \frac{\Delta t}{2}\boldsymbol{c} + \frac{\Delta t^2}{4}\boldsymbol{k}\right)\ddot{\boldsymbol{u}}_{i+1} = \boldsymbol{f}_{i+1} - \boldsymbol{c}\left(\dot{\boldsymbol{u}}_i + \frac{\Delta t}{2}\ddot{\boldsymbol{u}}_i\right) - \boldsymbol{k}\left(\boldsymbol{u}_i + \Delta t\,\dot{\boldsymbol{u}}_i + \frac{\Delta t^2}{4}\ddot{\boldsymbol{u}}_i\right)$$

$$\dot{\boldsymbol{u}}_{i+1} = \dot{\boldsymbol{u}}_i + \frac{\Delta t}{2}(\ddot{\boldsymbol{u}}_i + \ddot{\boldsymbol{u}}_{i+1})$$

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta t\,\dot{\boldsymbol{u}}_i + \frac{\Delta t^2}{4}(\ddot{\boldsymbol{u}}_i + \ddot{\boldsymbol{u}}_{i+1})$$

3. Newmark's linear acceleration method

The acceleration is assume to be linear over the interval

- Numerically updates from t_i to t_{i+1}

At time t_i, the acceleration, velocity and displacement are known. The force is prescribed.

$$\left(m + \frac{\Delta t}{2} c + \frac{\Delta t^2}{6} k\right) \ddot{u}_{i+1} = f_{i+1} - c\left(\dot{u}_i + \frac{\Delta t}{2} \ddot{u}_i\right) - k\left(u_i + \Delta t\, \dot{u}_i + \frac{\Delta t^2}{3} \ddot{u}_i\right)$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta t\, \ddot{u}_i + \frac{\Delta t^2}{2\,\Delta t} (\ddot{u}_{i+1} - \ddot{u}_i) = \dot{u}_i + \frac{\Delta t}{2} (\ddot{u}_i + \ddot{u}_{i+1})$$

$$u_{i+1} = u_i + \Delta t\, \dot{u}_i + \frac{\Delta t^2}{2} \ddot{u}_i + \frac{\Delta t^3}{6\,\Delta t} (\ddot{u}_{i+1} - \ddot{u}_i) = u_i + \Delta t\, \dot{u}_i + \frac{\Delta t^2}{6} (2\,\ddot{u}_i + \ddot{u}_{i+1})$$

- For Multiple degree of freedom systems

$$\left(\boldsymbol{m} + \frac{\Delta t}{2} \boldsymbol{c} + \frac{\Delta t^2}{6} \boldsymbol{k}\right) \ddot{\boldsymbol{u}}_{i+1} = \boldsymbol{f}_{i+1} - \boldsymbol{c}\left(\dot{\boldsymbol{u}}_i + \frac{\Delta t}{2} \ddot{\boldsymbol{u}}_i\right) - \boldsymbol{k}\left(\boldsymbol{u}_i + \Delta t\, \dot{\boldsymbol{u}}_i + \frac{\Delta t^2}{3} \ddot{\boldsymbol{u}}_i\right)$$

$$\dot{\boldsymbol{u}}_{i+1} = \dot{\boldsymbol{u}}_i + \frac{\Delta t}{2} (\ddot{\boldsymbol{u}}_i + \ddot{\boldsymbol{u}}_{i+1})$$

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta t\, \dot{\boldsymbol{u}}_i + \frac{\Delta t^2}{6} (2\,\ddot{\boldsymbol{u}}_i + \ddot{\boldsymbol{u}}_{i+1})$$

4. General Newmark's method

$$\boldsymbol{m}\,\ddot{\boldsymbol{u}} + \boldsymbol{c}\,\dot{\boldsymbol{u}} + \boldsymbol{k}\,\boldsymbol{u} = \boldsymbol{f}(t); \quad \boldsymbol{u}(0) = \boldsymbol{u}_0; \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$$

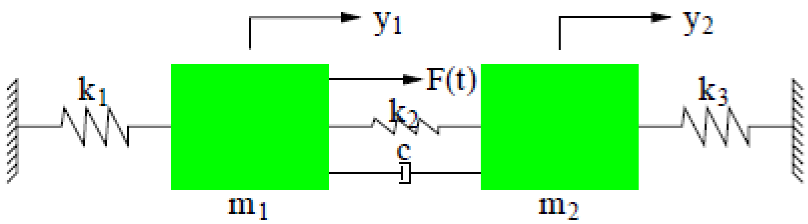
$$\dot{\boldsymbol{u}}_{i+1} = \dot{\boldsymbol{u}}_i + (1 - \gamma)\,\Delta t\, \ddot{\boldsymbol{u}}_i + \gamma \Delta t\, \ddot{\boldsymbol{u}}_{i+1}$$

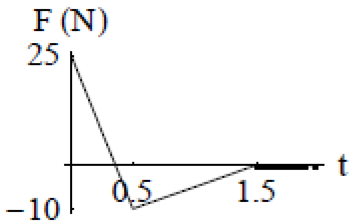
$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta t\, \dot{\boldsymbol{u}}_i + \left(\frac{1}{2} - \beta\right) \Delta t^2\, \ddot{\boldsymbol{u}}_i + \beta\, \Delta t^2\, \ddot{\boldsymbol{u}}_{i+1}$$

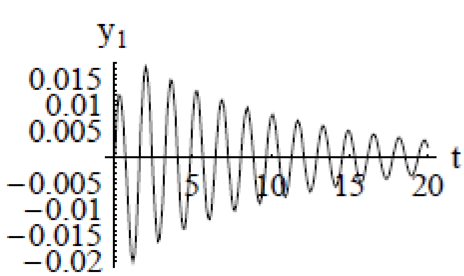
Solution procedure in the incremental form

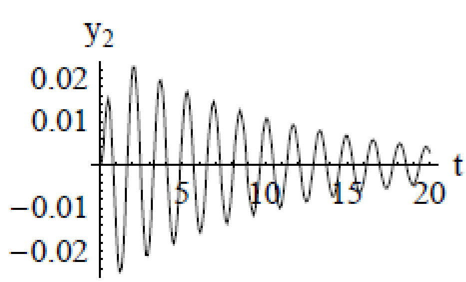
Solution algorithm

5. Example


y1
y2
k1
F(t)
k2
c
k3
m1
m2

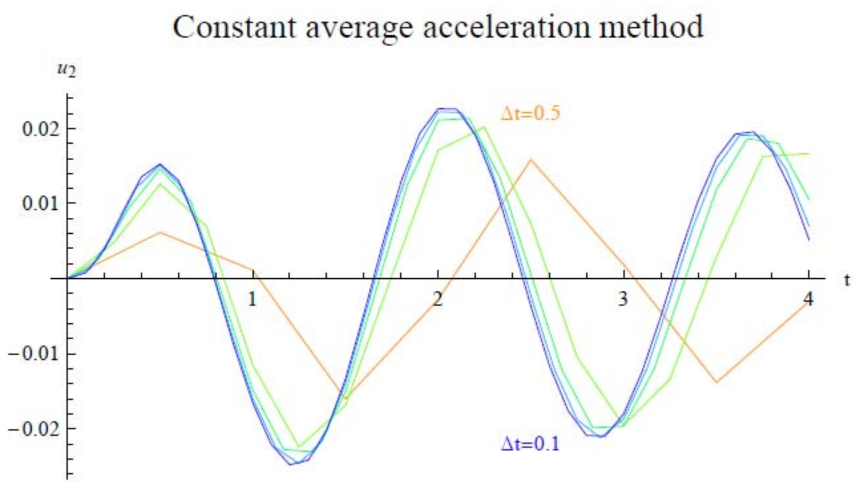

F (N)
25
−10
0.5
1.5
t


y1
0.015
0.01
0.005
−0.005
−0.01
−0.015
−0.02
5
10
15
20
t

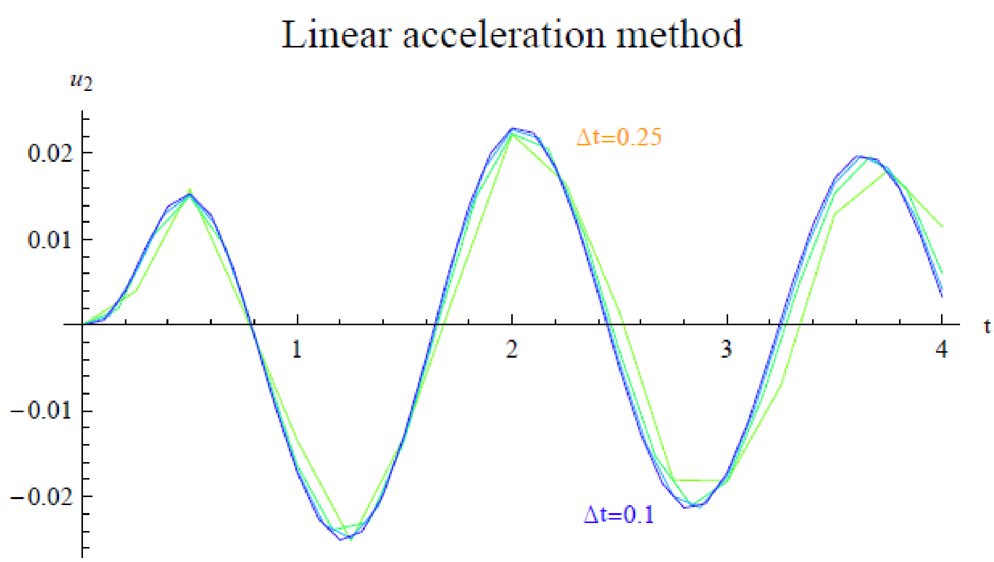

y2
0.02
0.01
−0.01
−0.02
5
10
15
20
t

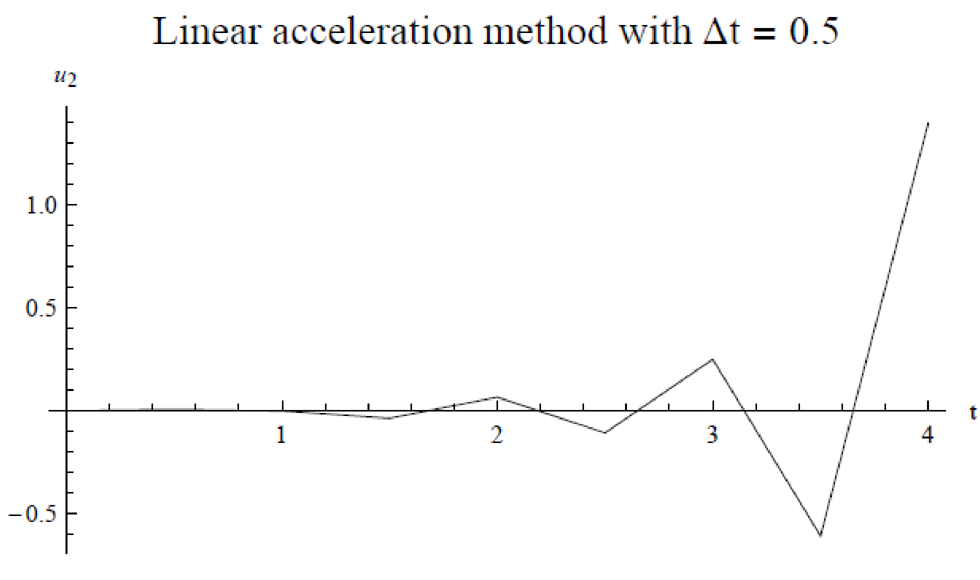
6. Stability

Constant average acceleration method: Unconditionally stable. Use a step size based on a trade-off between the desired accuracy and computational effort.



Linear acceleration method: For single degree of freedom system, solution is stable when $\Delta t = T/10$.





7. Finite difference method (Optional)

Approximation to derivatives

- Central difference method for SDOF systems

- Example

- Finite difference method for multidegree of freedom systems

8. Runge-Kutta Method for SDOF systems (Optional)

- Runge-Kutta method

- Example