Lecture 15: Determination of Natural Frequencies and Mode Shapes (Optional)

1. Eigenvalue problems

- The following type of equations often occur in practice,

  $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ (a)

  For a matrix of order N, there are N vectors $\mathbf{x}_i$ (i=1 to N). Every vector is associated with a value λ_i

  $\mathbf{x}_i$: Eigenvectors or Characteristic vectors

  λ_i: Eigenvalues

- Theoretical analysis

  Solving the following characteristic equation to obtain the eigenvalues

  $\det(\mathbf{A}-\lambda\mathbf{I})=0$

  Solving the following linear algebra equations to obtain the eigenvectors

  $(\mathbf{A}-\lambda_i\mathbf{I})\,\mathbf{x}_i=0$

- For vibrating system

  Solving the following characteristic equation to obtain the natural frequencies

  $\det(\mathbf{m}^{-1}\mathbf{k}-\lambda\mathbf{I})=0$

  or

  $\det(\mathbf{k}-\lambda\,\mathbf{m})=0$

$\omega_i = \text{sqrt}(\lambda_i)$

Solving the following linear algebra equations to obtain the eigenvectors

$(\mathbf{m}^{-1}\mathbf{k} - \lambda_i \mathbf{I})\, \boldsymbol{\varphi}_i = 0$

or

$(\mathbf{k} - \lambda_i\, \mathbf{m})\, \boldsymbol{\varphi}_i = 0$

2. Eigenvectors by gauss elimination

Example:

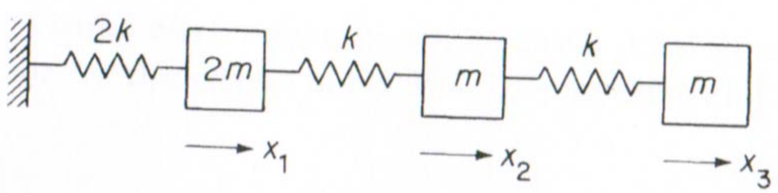

Figure 8.1.1.

3. Vector iteration (Power method) for the largest eigenvalue

$$\begin{bmatrix} 16 & -24 & 18 \\ 3 & -2 & 0 \\ -9 & 18 & -17 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \lambda \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

a). Guess a solution: $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

b). Substitute the guessed solution into LHR of the equation. Then, normalizing the resulting vector

$$\begin{bmatrix} 16 & -24 & 18 \\ 3 & -2 & 0 \\ -9 & 18 & -17 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 1 \\ -8 \end{Bmatrix} = 10 \begin{Bmatrix} 1.0 \\ 0.1 \\ -0.8 \end{Bmatrix}$$

c). Do iteration till convergence

Illustrating convergence towards the eigenvalue –8 and eigenvector {-0.5, 0.25, 1}

4. Calculation of intermediate eigenvalues - deflation

Using orthogonality of eigenvectors, a modified matrix **A*** can be established if the largest eigenvalue λ_1 and its corresponding eigenvector $\mathbf{x}_1$ are known.

$$\mathbf{A^*}=\mathbf{A}- \lambda_1 \mathbf{x}_1 (\mathbf{x}_1)^T$$

The power method can be employed to obtain the largest eigenvalue of **A***, which is the second largest eigenvalue of **A.**

Proof:

Example

Using iterative methods to find eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

5. Transformation methods

$$\mathbf{Ax} = \lambda\, \mathbf{x} \quad \Rightarrow \quad \mathbf{P}^{-1}\mathbf{APx} = \lambda\, \mathbf{P}^{-1}\, \mathbf{P}\, \mathbf{x}$$

For an orthogonal matrix **P**, $\mathbf{P}^{-1} = \mathbf{P}^{T}$

$$\mathbf{P}^{T}\mathbf{APx} = \lambda\, \mathbf{P}^{T}\, \mathbf{P}\, \mathbf{x} \Rightarrow \mathbf{A^*x} = \lambda\, \mathbf{x}$$

If **A** is symmetrical, $\mathbf{A^*}=\mathbf{P}^{T}\mathbf{AP}$ is also symmetrical.

In this course, we only consider **A** as a 2x2 matrix.

Select a 'rotation matrix' as **P,**

$$\mathbf{P}=\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

With a proper α, **A*** can be written as a diagonal matrix. Then, eigenvalues can be determined.

Example:

6. Dunkerley's formula

- Dunkerley's formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts.

- Flexibility matrix

- Dunkerley's formula

Example

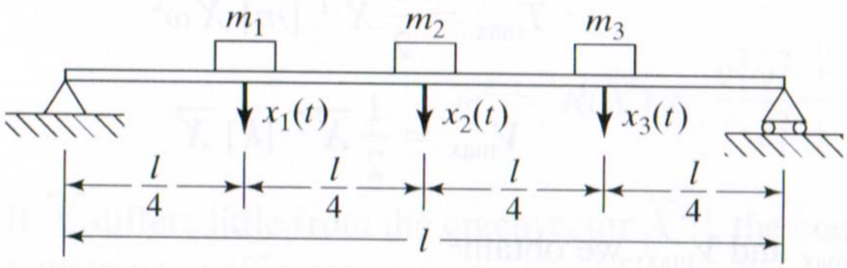

FIGURE 7.1 Beam carrying masses.

7. Rayleigh's method

- Rayleigh principle:

The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode. This stationary value, in fact, is a minimum value in the neighborhood of the fundamental natural mode.

- Rayleigh's method

- The above equation can be used to find an approximate value of the first natural frequency of the system. For this, we select a trial vector **X** to represent the first natural mode $\mathbf{X}^{(1)}$ and substitute it on the right hand side of the above equation. This yields the approximate value of ω_1^2. Because Rayleigh's quotient is stationary, remarkably good estimates of ω_1^2 can be obtained even if the trail vector **X** deviates greatly from the true natural mode $\mathbf{X}^{(1)}$. Obviously, the estimated value of the fundamental frequency ω_1 is more accurate if the trail vector **X** chosen resembles the true natural mode $\mathbf{X}^{(1)}$ closely.

Example