Lecture 13: Frequency Domain Solution

Reading materials: Sections 4.6 and 4.7

1. Steady-state solution for complex forcing function

- Equation of motion (Single degree of freedom)

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = F\,e^{i\,\Omega\,t}; \quad u(0) = u_0 \quad \dot{u}(0) = \dot{u}_0$$

- For steady-state solution, assume the solution

$$u(t) = U\,e^{i\,\Omega\,t}$$

- Get the solution

$$-\Omega^2\,m\,U\,e^{i\,\Omega\,t} + i\;\Omega\,c\,U\,e^{i\,\Omega\,t} + k\,U\,e^{i\,\Omega\,t} = F\,e^{i\,\Omega\,t}$$

$$U = \left[-\Omega^2\,m\; + \;i\;\Omega\,c\; + k\right]^{-1} F$$

$$\dot{u}(t) = i\,\Omega\,U\;\;e^{i\,\Omega\,t} = \dot{U}\,e^{i\,\Omega\,t} \quad \Longrightarrow \quad \dot{U} \equiv i\,\Omega\,U$$

$$\ddot{u}(t) = -\,\Omega^2\,U\;\;e^{i\,\Omega\,t} = \ddot{U}\,e^{i\,\Omega\,t} \quad \Longrightarrow \quad \ddot{U} \equiv -\,\Omega^2\,U$$

Not time derivatives.

- For Periodic loading, using Fourier transform

$$\boxed{f(t) = \sum_{n=-p}^{p} F_n\,e^{i\,n\,\Omega\,t}}$$

$$\boxed{F_n = \frac{\Omega}{2\pi}\int_0^T f(t)\,e^{-i\,n\,\Omega\,t}\,\mathrm{dt}; \quad n = 0,\,1,\,2,\,\ldots\,; \quad F_{-n} = \mathrm{Conj}(F_n)}$$

- For each component, the solution amplitude is

$$U_n = \left[-n^2\,\Omega^2\,m \;+\; i\;n\,\Omega\,c\;+k\right]^{-1} F_n \equiv H_n\,F_n$$

$$H_n = \frac{1}{-n^2\,\Omega^2\,m + i\;n\,\Omega\,c + k}$$

here

Natural frequency, $\omega = \sqrt{k/m} \quad \Longrightarrow \quad \frac{m}{k} = \frac{1}{\omega^2}$

Damping ratio, $\xi = \frac{c}{2\,m\,\omega} \quad \Longrightarrow \quad \frac{c}{k} = \frac{2\,m\,\omega\,\xi}{k} = \frac{2\,\xi}{\omega}$

Frequency ratio, $r_n = \frac{n\,\Omega}{\omega}$

Therefore,

$$H_n = \frac{1}{k\left(-r_n^2 + 2\,i\;r_n\,\xi + 1\right)}$$

- Complete solution

$$u(t) = \sum_{n=-p}^{p} U_n\,e^{i\,n\,\Omega\,t}$$

- Summary of the procedure

  a. Use the direct Fourier transform to determine the complex frequency amplitudes for the applied force – transform force into the frequency domain

$$F_n = \frac{\Omega}{2\pi}\int_0^T f(t)\,e^{-i\,n\,\Omega\,t}\,\mathrm{dt}; \quad n = 0,\,1,\,2,\,\ldots\,; \quad F_{-n} = \mathrm{Conj}(F_n)$$

b. Determine the complex frequency amplitudes of the solution

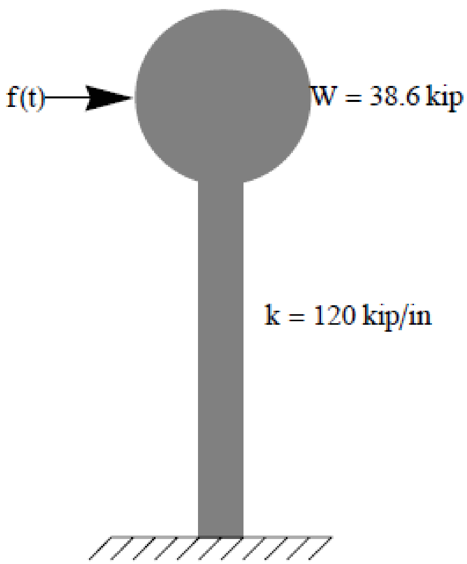
$$H_n = \frac{1}{-n^2\,\Omega^2\,m + i\,n\,\Omega\,c + k} = \frac{1}{k\left(-r_n^2 + 2\,i\,r_n\,\xi + 1\right)}$$

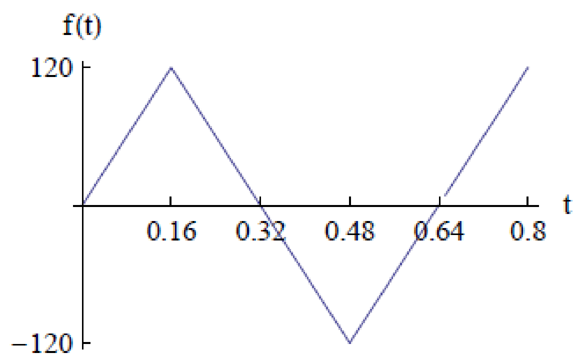
$$U_n = H_n\,F_n; \quad \dot{U}_n = i\,n\,\Omega\,H_n\,F_n; \quad \ddot{U}_n = -n^2\,\Omega^2\,H_n\,F_n$$

c. Use the inverse Fourier transform to take the frequency amplitudes of the solution back into the time domain

$$u(t) = \sum_{n=-p}^{p} U_n\,e^{i\,n\,\Omega\,t}; \quad \dot{u}(t) = \sum_{n=-p}^{p} i\,n\,\Omega\,U_n\,e^{i\,n\,\Omega\,t}; \quad \ddot{u}(t) = -\sum_{n=-p}^{p} n^2\,\Omega^2\,U_n\,e^{i\,n\,\Omega\,t}$$

Example: Determine the steady-state motion of the water tower when it is subjected to a triangular periodic load. 10% damping





$$\text{Load period, } T = 0.64 \implies \Omega = \frac{2\pi}{0.64} = 9.81748$$

$$f(t) = \begin{cases} 750.\,(t-0.16)+120 & t \le 0.16 \\ -750.\,(t-0.48)-120 & t \le 0.48; \\ 750.\,(t-0.64) & t \le 0.64 \end{cases} \qquad \Omega = 9.81748$$

$m = 0.1;$ $k = 120.;$ $\omega = 34.641$

$\xi = 0.1;$ $c = 0.69282;$

$F_{-3} = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(-3)(9.81748)t}\, dt = -5.4038\, i;$ $H_{-3} = 0.0218456 + 0.0134042\, i;$ $U_{-3} = 0.0724338 - 0.118049\, i$

$F_{-2} = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(-2)(9.81748)t}\, dt = 0;$ $H_{-2} = 0.0119447 + 0.00199504\, i;$ $U_{-2} = 0$

$F_{-1} = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(-1)(9.81748)t}\, dt = 48.6342\, i;$ $H_{-1} = 0.00902683 + 0.000556336\, i;$ $U_{-1} = -0.0270569 + 0.439012\, i$

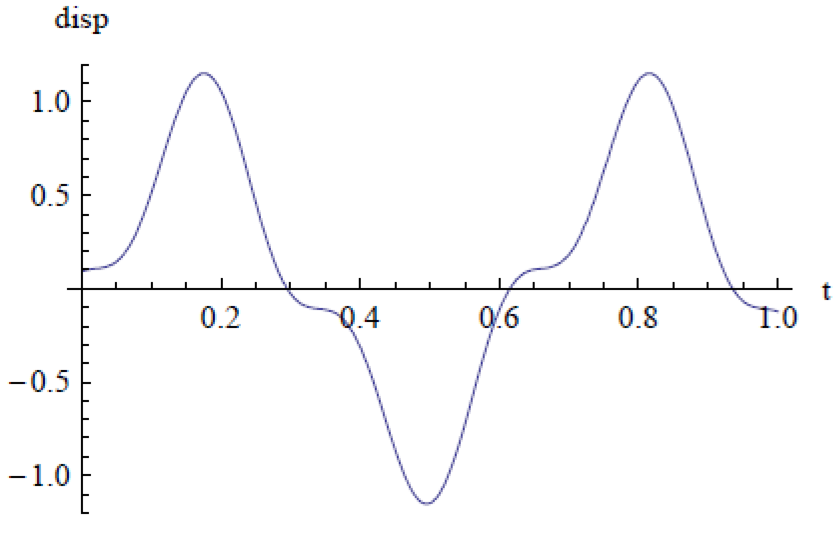
$F_0 = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(0)(9.81748)t}\, dt = 0;$ $H_0 = 0.00833333;$ $U_0 = 0$

$F_1 = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(1)(9.81748)t}\, dt = -48.6342\, i;$ $H_1 = 0.00902683 - 0.000556336\, i;$ $U_1 = -0.0270569 - 0.439012\, i$

$F_2 = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(2)(9.81748)t}\, dt = 0;$ $H_2 = 0.0119447 - 0.00199504\, i;$ $U_2 = 0$

$F_3 = (1/0.64)\int_0^{0.64} f(t)\, e^{(-i)(3)(9.81748)t}\, dt = 5.4038\, i;$ $H_3 = 0.0218456 - 0.0134042\, i;$ $U_3 = 0.0724338 + 0.118049\, i$

$$u(t) = (-0.0270569 + 0.439012\, i)\, e^{-9.81748\, i\, t} - (0.0270569 + 0.439012\, i)\, e^{9.81748\, i\, t}$$
$$+ (0.0724338 - 0.118049\, i)\, e^{-29.4524\, i\, t} + (0.0724338 + 0.118049\, i)\, e^{29.4524\, i\, t}$$



2. Using discrete Fourier Transform for numerical force data

- Use the direct Discrete Fourier Transform to compute the complex frequency amplitudes for the applied force

$$F_n = \frac{1}{N}\sum_{s=0}^{N-1} f_s\, e^{-i(2\pi\, s\, n/N)}; \quad n = 0, 1, 2, \ldots, N-1$$

- Determine the complex frequency amplitudes of the solution

$$H_n = \frac{1}{-n^2\,\Omega^2\, m + i\, n\,\Omega\, c + k}$$

$$U_n = H_n\, F_n; \quad \dot{U}_n = i\, n\, \Omega\, H_n\, F_n; \quad \ddot{U}_n = -n^2\,\Omega^2\, H_n\, F_n$$

- Use the inverse Fourier transform to transform frequency amplitudes of the solution back into the time domain

$$u_s = \sum_{n=0}^{N-1} U_n\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

$$\dot{u}_s = \sum_{n=0}^{N-1} \dot{U}_n\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

$$\ddot{u}_s = \sum_{n=0}^{N-1} \ddot{U}_n\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

3. General Non-Periodic loading

- General loading can be treated in the frequency domain approach by extending the load period to include a large interval of zero force to the end of the actual force. Mathematically the load is still periodic.

- The method is used only for large duration loading where the initial built-up of the loading is relatively slow.

4. Impulsive force

- A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period of time and then stops.

- The simplest form is the impulsive force – a force that has a large magnitude F and acts for a very short period of time.

5. Multiple degrees of freedom frequency domain solution

- Coupled Equation of motion

$$\boldsymbol{m}\,\ddot{\boldsymbol{u}} + \boldsymbol{c}\,\dot{\boldsymbol{u}} + \boldsymbol{k}\,\boldsymbol{u} = \boldsymbol{f}(t)$$

Rayleigh damping

$$\boldsymbol{c} = \alpha\,\boldsymbol{m} + \beta\,\boldsymbol{k}$$

- Undamped free vibration modes

$$\boldsymbol{k}\,\boldsymbol{\phi}_i = \lambda_i\,\boldsymbol{m}\,\boldsymbol{\phi}_i; \quad i = 1, 2, \ldots, n$$

Mass orthogonality:

$$\boldsymbol{\phi}_j^T\,\boldsymbol{m}\,\boldsymbol{\phi}_i = 0; \quad i \neq j$$

Stiffness orthogonality:

$$\boldsymbol{\phi}_j^T\,\boldsymbol{k}\,\boldsymbol{\phi}_i = 0; \quad i \neq j$$

- Uncoupled equations

Modal coordinates

$$\boldsymbol{z} = (\,z_1 \quad z_2 \quad \ldots \quad z_n\,)^T$$

$$\boldsymbol{u}(t) = \sum_i z_i(t)\,\boldsymbol{\phi}_i$$

Damped modal equations

$$M_i\,\ddot{z}_i(t) + (\alpha\,M_i + \beta\,K_i\,)\,\dot{z}_i(t) + K_i\,z_i(t) = R_i; \quad i = 1, 2, \ldots$$

$$z_i(0) = \frac{1}{M_i}\left(\phi_i^T\, m\, u^0\right); \quad \dot{z}_i(0) = \frac{1}{M_i}\left(\phi_i^T\, m\, v^0\right);$$

$$M_i = \phi_i^T\, m\, \phi_i; \quad K_i = \phi_i^T\, k\, \phi_i; \quad \omega_i = \sqrt{K_i / M_i}\,; \quad R_i = \phi_i^T\, F$$

$$C_i = \alpha\, M_i + \beta\, K_i$$

$$M_i\, \ddot{z}_i(t) + C_i\, \dot{z}_i(t) + K_i\, z_i(t) = R_i; \quad i = 1, 2, \ldots$$

Solution of modal equations

a. Use the direct Discrete Fourier transform to compute the complex frequency amplitudes for the applied force

$$F_{i,n} = \frac{1}{N} \sum_{s=0}^{N-1} R_{i,s}\, e^{-i(2\pi\, s\, n/N)}\,; \quad n = 0, 1, 2, \ldots, N-1$$

b. Determine the complex frequency amplitudes of the solution

$$H_{i,n} = \frac{1}{-n^2\, \Omega^2\, M_i + i\, n\, \Omega\, C_i + K_i}$$

$$Z_{i,n} = H_{i,n}\, F_{i,n}; \quad \dot{Z}_{i,n} = i\, n\, \Omega\, H_{i,n}\, F_{i,n}; \quad \ddot{Z}_{i,n} = -n^2\, \Omega^2\, H_{i,n}\, F_{i,n}$$

c. Use the Inverse Fourier transform to transform frequency amplitudes of the solution back into the time domain

$$z_{i,s} = \sum_{n=0}^{N-1} Z_{i,n}\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

$$\dot{z}_{i,s} = \sum_{n=0}^{N-1} \dot{Z}_{i,n}\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

$$\ddot{z}_{i,s} = \sum_{n=0}^{N-1} \ddot{Z}_{i,n}\, e^{i\,(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots, N$$

Final solution

$$u = \sum_i z_i\, \phi_i$$