Lecture 12: Discrete Fourier Transform (Optional)

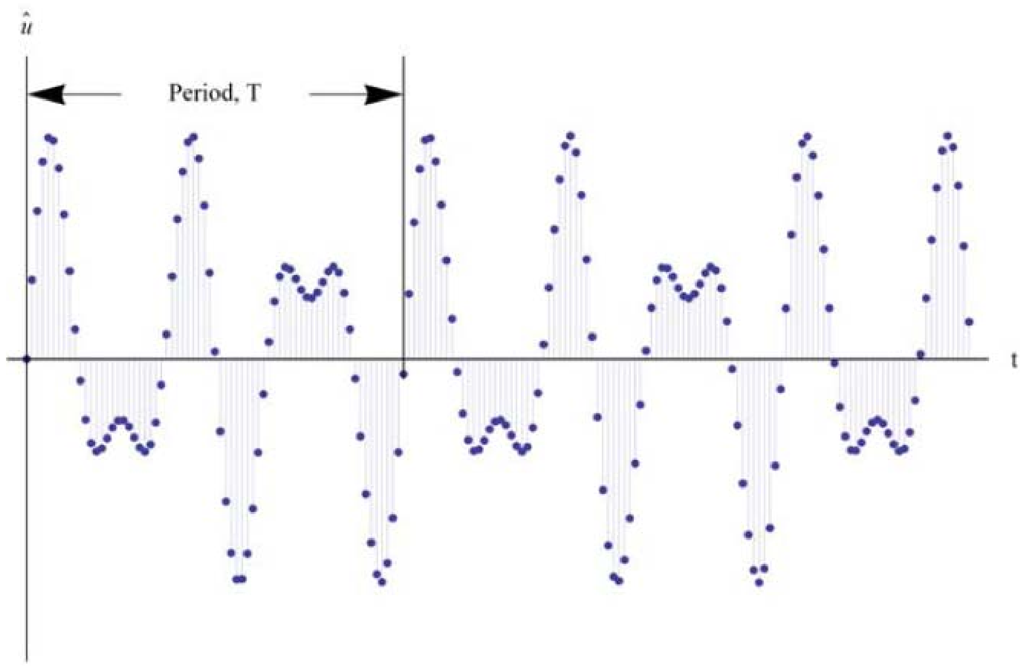
Reading materials: Section 4.5

1. Introduction

- All experimental vibration time signals are available as numerical data at discrete time steps.

- For actual dynamic loading, including wind or earthquakes, the results can only be obtained through numerical integration of equations of motion.

- It is important to create a discrete form of the Fourier series that can be used on discrete time signals.



- The complex Fourier series

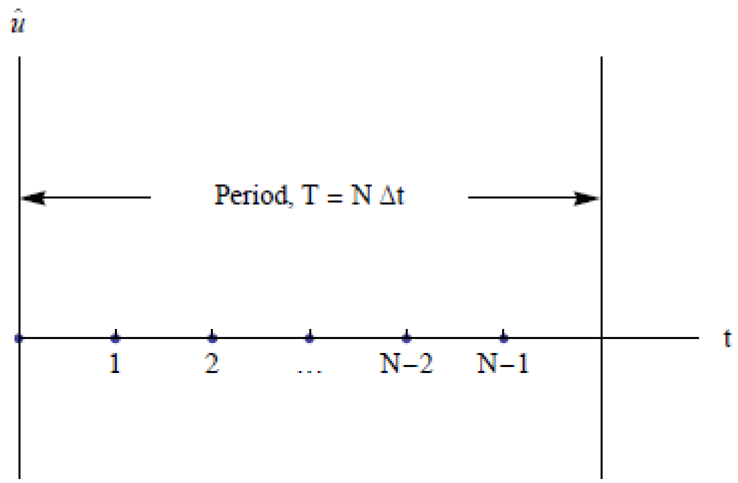
$$U_n = \frac{1}{T}\int_0^T \hat{u}(t)\, e^{-i n \omega t}\, \mathrm{dt}; \quad n = 0, 1, 2, \ldots, p\,; \quad U_{-n} = \mathrm{Conj}(U_n)$$

$$u(t) = \sum_{n=-p}^{p} U_n\, e^{i n \omega t}$$

Period and frequency

$$\Delta t = \frac{T}{N}; \quad \Longrightarrow \quad T = N\,\Delta t; \quad \Longrightarrow \quad \omega = \frac{2\pi}{N\,\Delta t}$$

Numerical integration (Trapezoidal rule)



$$\Delta t = \frac{T}{N}; \quad \Longrightarrow \quad T = N\,\Delta t; \quad \Longrightarrow \quad \omega = \frac{2\pi}{N\,\Delta t}$$

$$U_n = \frac{1}{T}\int_0^T \hat{u}(t)\, e^{-i n \omega t}\, dt;$$

$$U_n = \frac{1}{T}\Big[\frac{\Delta t}{2}\left(\hat{u}_0\, e^{-i n \omega t_0} + \hat{u}_1\, e^{-i n \omega t_1}\right) + \frac{\Delta t}{2}\left(\hat{u}_1\, e^{-i n \omega t_1} + \hat{u}_2\, e^{-i n \omega t_2}\right) + \ldots\Big]$$

$$U_n = \frac{\Delta t}{T}\Big[\frac{1}{2}\,\hat{u}_0\, e^{-i n \omega t_0} + \hat{u}_1\, e^{-i n \omega t_1} + \hat{u}_2\, e^{-i n \omega t_2} + \ldots + \hat{u}_{N-1}\, e^{-i n \omega t_{N-1}} + \frac{1}{2}\,\hat{u}_N\, e^{-i n \omega t_N}\Big]$$

$$U_n = \frac{\Delta t}{T}\sum_{s=0}^{N-1} \hat{u}_s\, e^{-i n \omega t_s}; \quad n = 0, \pm 1, \pm 2, \ldots, \pm p$$

Here, $t_S = s\Delta t$,

$$\omega\, t_s = \frac{2\pi}{N\,\Delta t}\, s\,\Delta t = \frac{2\pi s}{N} \text{ and } \frac{\Delta t}{T} = \frac{1}{N}$$

$$U_n = \frac{1}{N}\sum_{s=0}^{N-1} \hat{u}_s\, e^{-i(2\pi\, s\, n/N)}\,; \quad n = 0,\, \pm 1,\, \pm 2,\, \ldots$$

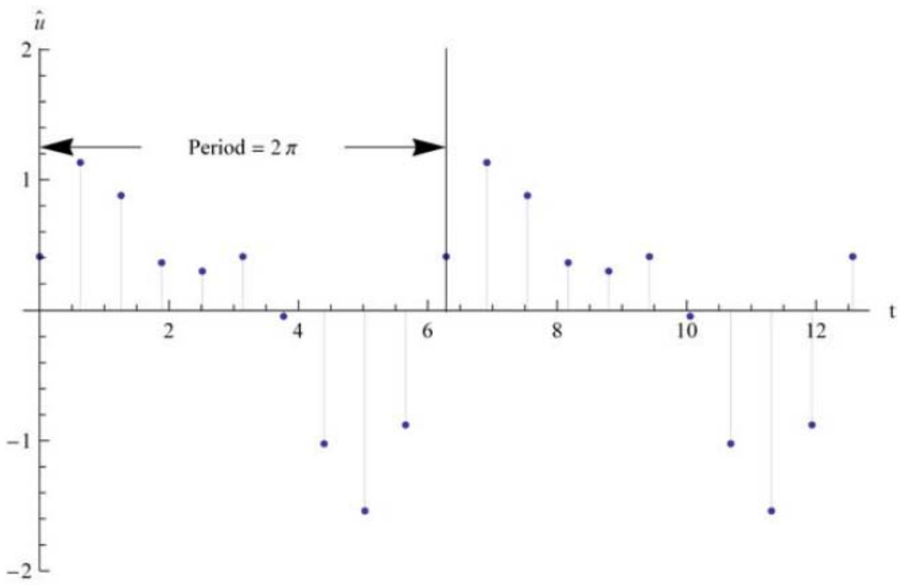
$$n\,\omega = \frac{2\pi n}{N\,\Delta t}; \quad n = 0,\, \pm 1,\, \pm 2,\, \ldots,\, \pm p$$

In summary,

$$U_n = \frac{1}{N}\sum_{s=0}^{N-1} \hat{u}_s\, e^{-i(2\pi\, s\, n/N)}\,; \quad n = 0,\, \pm 1,\, \pm 2,\, \ldots,\, \pm p$$

$$u_s = \sum_{n=-p}^{p} U_n\, e^{i(2\pi\, s\, n/N)}; \quad s = 0,\, 1,\, 2,\, \ldots,\, N-1$$

2. Example



T = 6.28319 s; ω = 1.; N = 10; Δt = 0.628319

s	t	$\hat{u}_s$
0	0.	0.408983
1	0.628319	1.13169
2	1.25664	0.878228
3	1.88496	0.362137
4	2.51327	0.296642
5	3.14159	0.408983
6	3.76991	−0.043876
7	4.39823	−1.02389
8	5.02655	−1.53998
9	5.65487	−0.878929

$$U_n = \frac{1}{N} \sum_{s=0}^{N-1} \hat{u}_s \, e^{-i(2\pi\, s\, n/N)}; \quad n = 0, \pm 1, \pm 2, \ldots$$

$U_{-2} = 0.204492 + 0.219507\, i$

$U_{-1} = 0.5\, i$

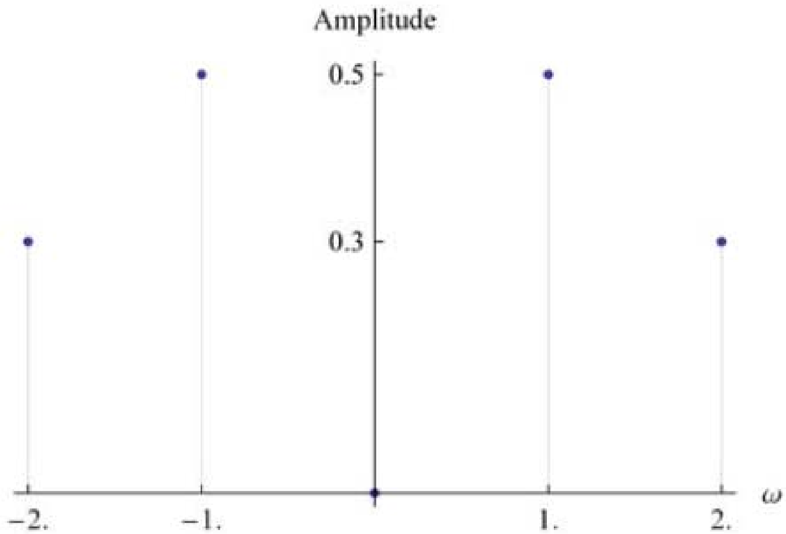
$U_0 = 0$

$U_1 = -0.5\, i$

$U_2 = 0.204492 - 0.219507\, i$

$$n\,\omega = \frac{2\pi n}{N\,\Delta t}; \quad n = 0, \pm 1, \pm 2, \ldots$$

	−2	−1	0	1	2
Frequency	−2.	−1.	0	1.	2.
Amplitude	0.3	0.5	0	0.5	0.3



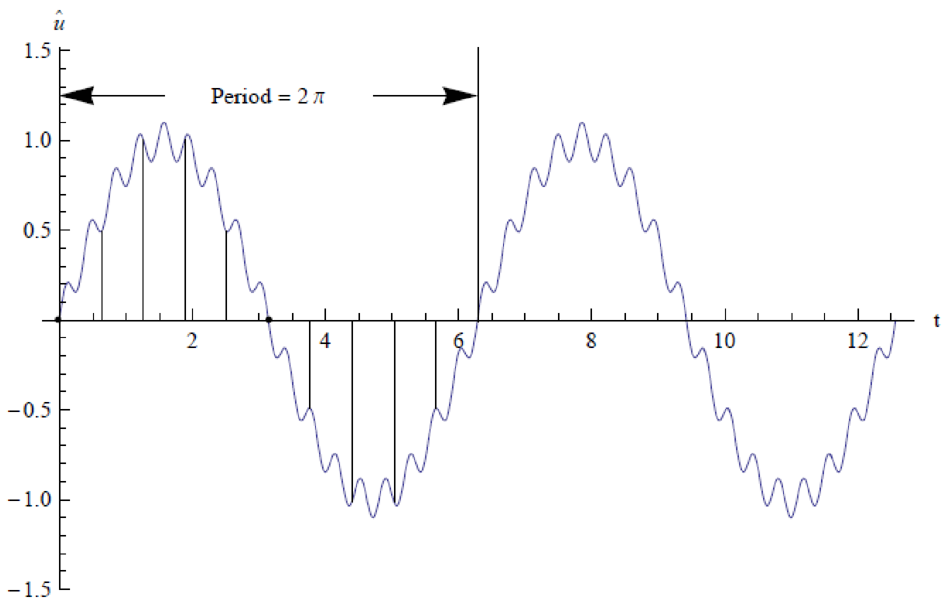
$$u_s = \sum_{n=-\infty}^{\infty} U_n\, e^{i(2\pi s n/N)}; \quad s = 0, 1, 2, \ldots$$

t	$\hat{u}$	u
0.	0.408983	0.408983
0.628319	1.13169	1.13169
1.25664	0.878228	0.878228
1.88496	0.362137	0.362137
2.51327	0.296642	0.296642
3.14159	0.408983	0.408983
3.76991	−0.043876	−0.043876
4.39823	−1.02389	−1.02389
5.02655	−1.53998	−1.53998
5.65487	−0.878929	−0.878929

3. Data smoothing

- Discrete Fourier transform can be used to remove noise from measured time-history data.

- An Example



- List of data over one period

$T = 6.28319$ s; $\omega = 1.$; $N = 10$; $\Delta t = 0.628319$

s	t	$\hat{u}_s$
0	0.	0.
1	0.628319	0.49268
2	1.25664	1.00984
3	1.88496	1.00984
4	2.51327	0.49268
5	3.14159	-2.46167×10^{-17}
6	3.76991	-0.49268
7	4.39823	-1.00984
8	5.02655	-1.00984
9	5.65487	-0.49268

Discrete Fourier frequency coefficients

$$U_n = \frac{1}{N} \sum_{s=0}^{N-1} u_s \, e^{-i(2\pi\, s\, n/N)}; \quad n = 0, 1, 2, \ldots; \quad U_{-n} = \text{Conj}\,(U_n)$$

$U_{-2} = 0$

$U_{-1} = 0.5\, i$

$U_0 = 0$

$U_1 = -0.5\, i$

$U_2 = 0$

	-2	-1	0	1	2
Frequency	$-2.$	$-1.$	0	1.	2.
Amplitude	0	0.5	0	0.5	0

Smoothed solution

$$u_s = \sum_{n=-\infty}^{\infty} U_n \, e^{i(2\pi\, s\, n/N)}; \quad s = 0, 1, 2, \ldots$$

t	$\hat{u}$	u
0.	0.	0
0.628319	0.49268	0.587785
1.25664	1.00984	0.951057
1.88496	1.00984	0.951057
2.51327	0.49268	0.587785
3.14159	-2.46167×10^{-17}	0
3.76991	−0.49268	−0.587785
4.39823	−1.00984	−0.951057
5.02655	−1.00984	−0.951057
5.65487	−0.49268	−0.587785