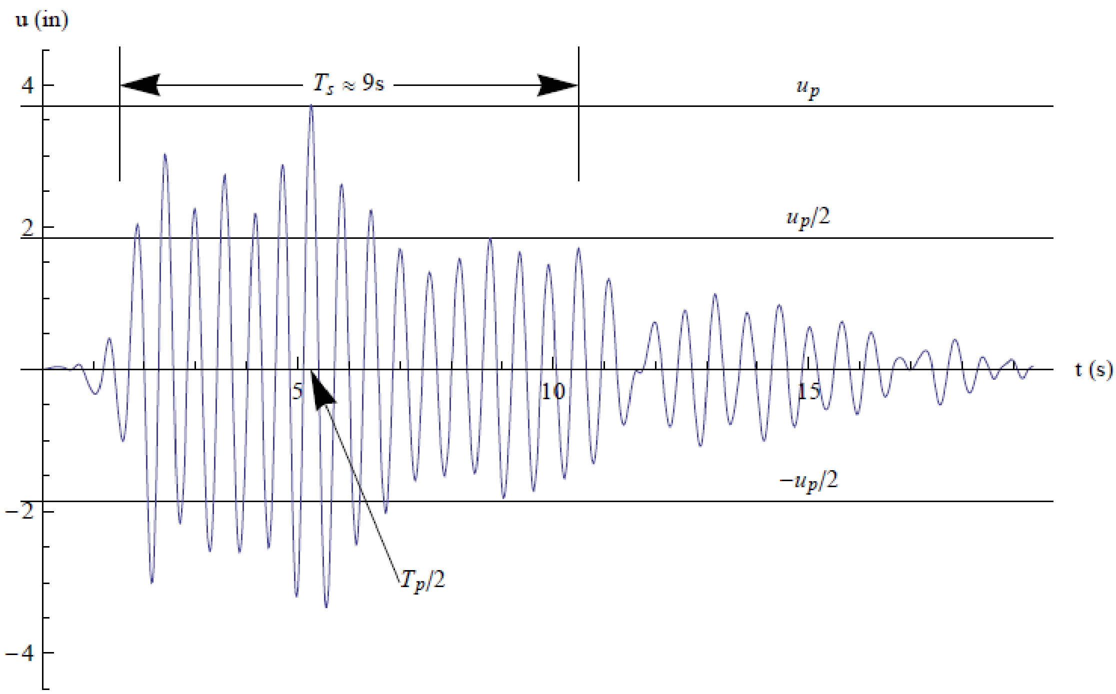
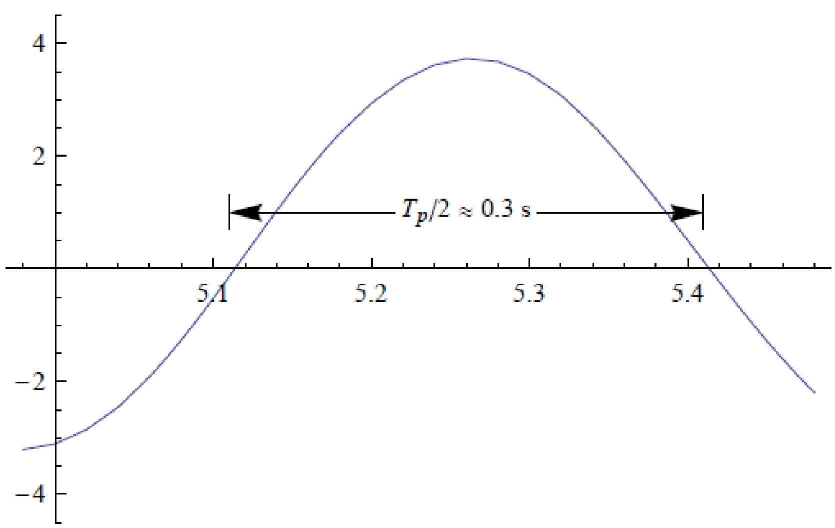
Lecture 11: Time frequency signals

Reading materials: Sections 4.1, 4.2, 4.3, and 4.4

1. Time domain vibration signal





The following can be obtained:

Duration of the record: 18.5 s

u_P=3.7, peak amplitude

T_P=0.6s, period in the neighborhood of the peak

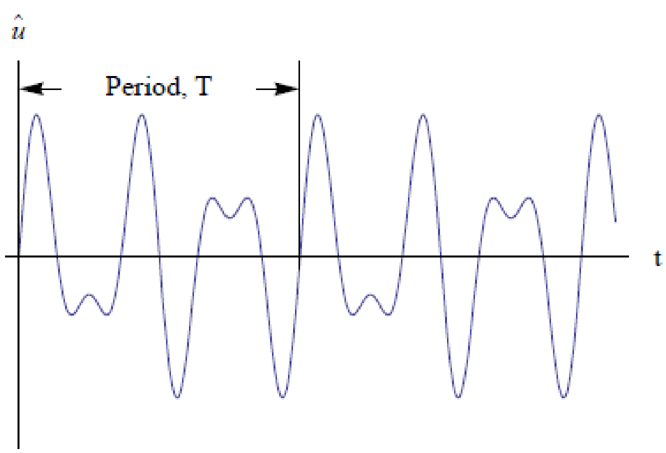
T_S=9 s, duration of strong motion

N_Z=30, number of zero crossings within T_S

2. Fourier series for periodic functions

If a time function repeats itself after T seconds, it is called a periodic function with period T.

$u(t) = u(t + T)$



A Fourier representation $u(t)$ of a given time function $\hat{u}(t)$

$$u(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega\, t + b_n \sin n\,\omega\, t)$$

Fourier coefficients:

$$a_0 = \frac{1}{T}\int_0^T \hat{u}(t)\,\mathrm{dt}$$

$$a_n = \frac{2}{T}\int_0^T \hat{u}(t)\cos n\,\omega\,t\,\mathrm{dt}; \quad n = 1, 2, \ldots$$

$$b_n = \frac{2}{T}\int_0^T \hat{u}(t)\sin n\,\omega\,t\,\mathrm{dt}; \quad n = 1, 2, \ldots$$

$$\text{Period} = T; \qquad \text{Frequency} = \omega = \frac{2\pi}{T}$$

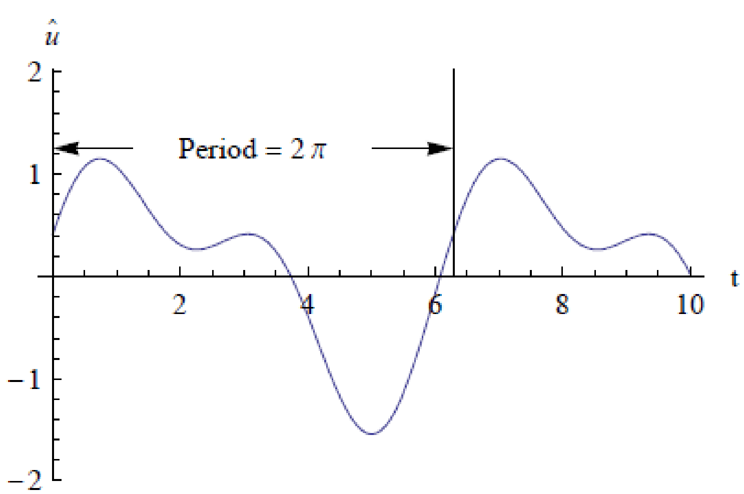
Two special classes of functions

Even function: $u(-t) = u(t)$, $b_n=0$

Odd function: $u(t) = -u(t)$, $a_0 = a_n = 0$

Example

$$\hat{u}(t) = \sin(t) + 0.6\sin(2t + 0.75)$$



3. Fourier spectrum

- Fourier series can be viewed in terms of a superposition of harmonics

$$u(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t)$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos n\,\omega\,t + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin n\,\omega\,t \right)$$

or

$$u(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\ (\cos\theta_n \cos n\,\omega\,t + \sin\theta_n \sin n\,\omega\,t)$$

$$u(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\ \cos(n\ \omega\,t - \theta_n)$$

Finally,

$$u(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\ \omega\,t - \theta_n)$$

$$A_n = \sqrt{a_n^2 + b_n^2}\ ; \quad \theta_n = \cos^{-1}\left(\frac{a_n}{A_n}\right) = \sin^{-1}\left(\frac{b_n}{A_n}\right)$$

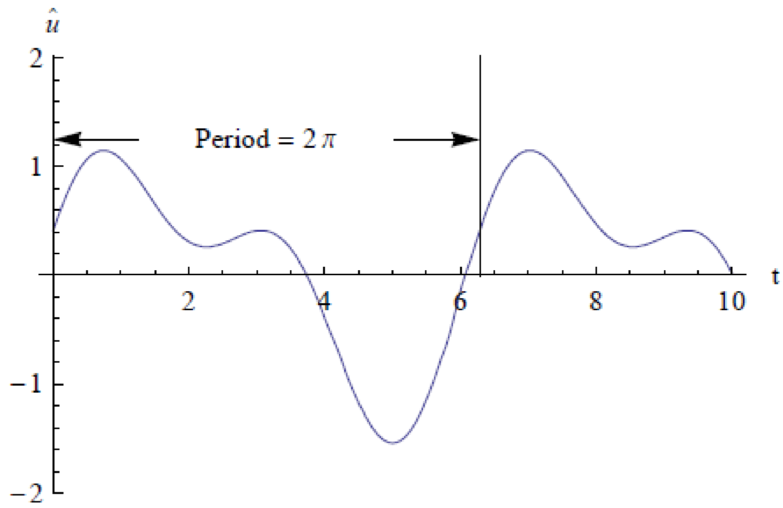
- The above form shows that the amplitude of each harmonic is A_n.

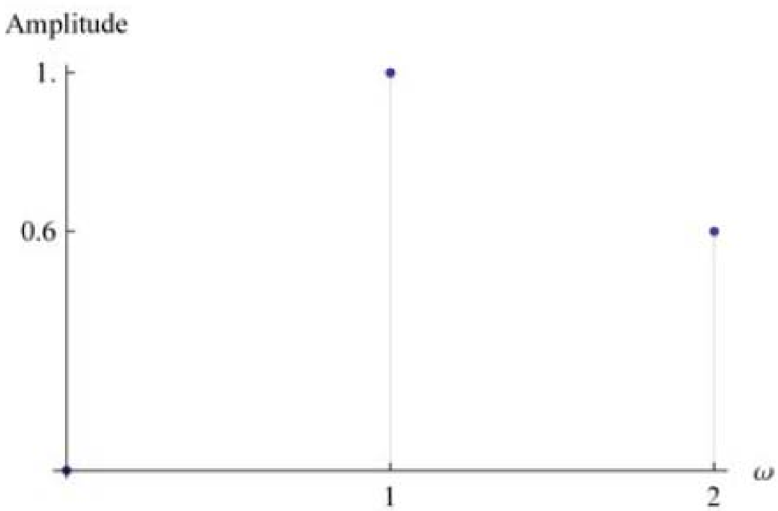
- A plot of the amplitudes for different frequencies is called the Fourier or frequency spectrum.

- This plot is typically used to show dominant frequencies and their amplitudes present in a given periodic function.

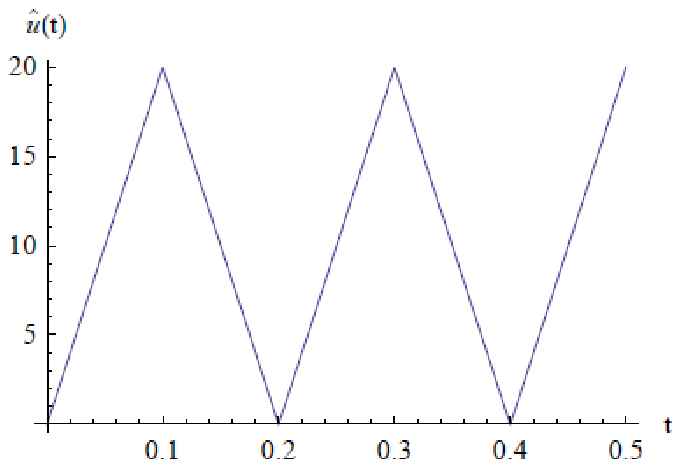
Example 1: Compute and draw frequency spectrum for the following function

$$\hat{u}(t) = \sin(t) + 0.6\sin(2t + 0.75)$$





Example 2: Compute and draw frequency spectrum for the following function



$$\hat{u}(t) = \begin{cases} 200\,t & \text{if} \quad t \le 0.1 \\ 40 - 200\,t & \text{if} \quad t > 0.1 \end{cases}$$

Period, T = 0.2; $\omega = 31.4159$

$$a_0 = (1/T)\int_0^{0.2} \hat{u}\, dt = 10.$$

$$a_1 = (2/T)\int_0^{0.2} \hat{u}\cos(31.4159t)dt = -8.10569$$

$$b_1 = (2/T)\int_0^{0.2} \hat{u}\sin(31.4159t)dt = 0$$

$$a_3 = (2/T)\int_0^{0.2} \hat{u}\cos(94.2478t)dt = -0.900633$$

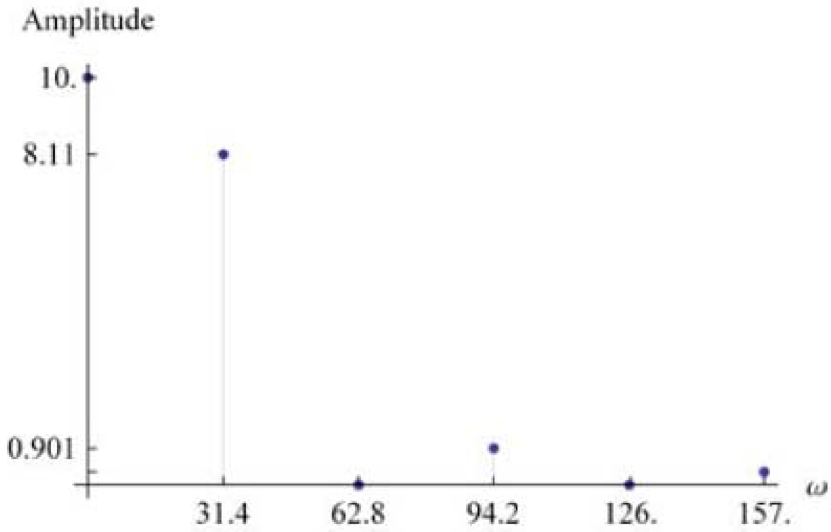
$$b_3 = (2/T)\int_0^{0.2} \hat{u}\sin(94.2478t)dt = 0$$

$$a_5 = (2/T)\int_0^{0.2} \hat{u}\cos(157.08t)dt = -0.324228$$

$$b_5 = (2/T)\int_0^{0.2} \hat{u}\sin(157.08t)dt = 0$$

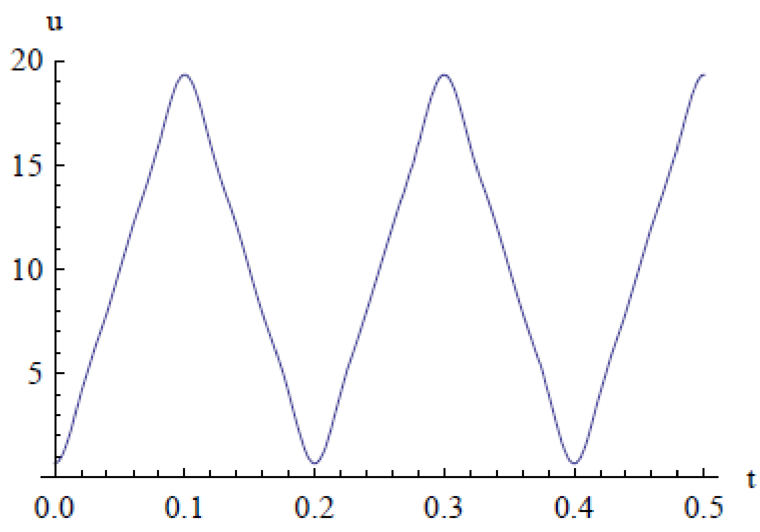
$$u(t) = -8.10569\cos(31.4159\,t) - 0.900633\cos(94.2478\,t) - 0.324228\cos(157.08\,t) + 10.$$

Frequency	0	31.4159	62.8319	94.2478	125.664	157.08
Amplitude	10.	8.10569	0	0.900633	0	0.324228



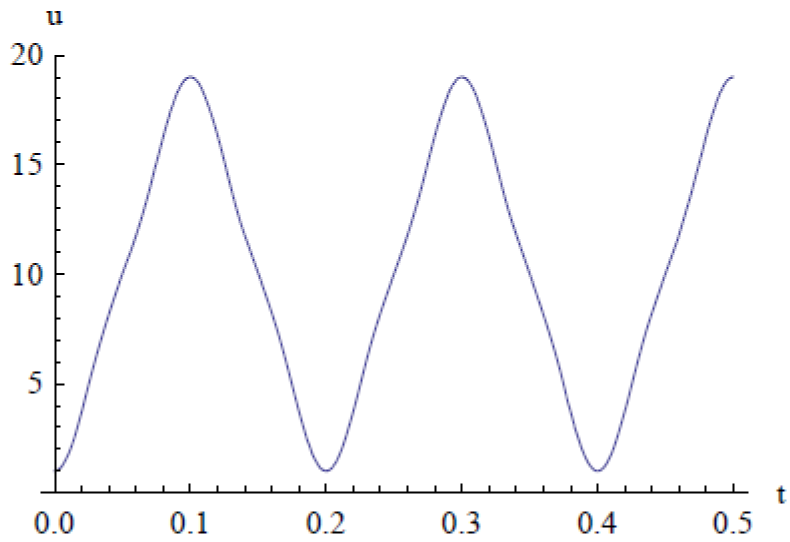
Fourier series representation

$$u(t) = -8.10569\cos(31.4159\,t) - 0.900633\cos(94.2478\,t) - 0.324228\cos(157.08\,t) + 10.$$



The frequency spectrum also suggests that the first and the third harmonics can give a reasonable representation of the original function.

$$u(t) = -8.10569\cos(31.4159\,t) - 0.900633\cos(94.2478\,t) + 10.$$



4. Complex Fourier series (Optional)

Fourier series of displacements in exponential form

$$u(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t)$$

$$u(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{e^{i\,n\,\omega\,t} + e^{-i\,n\,\omega\,t}}{2} + b_n \frac{e^{i\,n\,\omega\,t} - e^{-i\,n\,\omega\,t}}{2\,i}\right)$$

$$u(t) = a_0 + \sum_{n=1}^{\infty} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right) + \sum_{n=1}^{\infty} e^{-i\,n\,\omega\,t}\left(\frac{a_n}{2} - \frac{b_n}{2\,i}\right)$$

$$a_0 = \frac{1}{T}\int_0^T \hat{u}(t)\,\mathrm{dt}$$

$$a_n = \frac{2}{T}\int_0^T \hat{u}(t)\cos n\,\omega\,t\,\mathrm{dt} = \frac{1}{T}\int_0^T \hat{u}(t)\left(e^{i\,n\,\omega\,t} + e^{-i\,n\,\omega\,t}\right)\mathrm{dt}$$

$$b_n = \frac{2}{T}\int_0^T \hat{u}(t)\sin n\,\omega_f\,t\,\mathrm{dt} = \frac{1}{i\,T}\int_0^T \hat{u}(t)\left(e^{i\,n\,\omega\,t} - e^{-i\,n\,\omega\,t}\right)\mathrm{dt}$$

$$a_{-n} = a_n \quad \text{and} \quad b_{-n} = -b_n$$

$$\sum_{n=1}^{\infty} e^{-i\,n\,\omega\,t}\left(\frac{a_n}{2} - \frac{b_n}{2\,i}\right) = \sum_{n=-1}^{-\infty} e^{-i\,(-n)\,\omega\,t}\left(\frac{a_{-n}}{2} - \frac{b_{-n}}{2\,i}\right) = \sum_{n=-\infty}^{-1} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right)$$

$$u(t) = a_0 + \sum_{n=1}^{\infty} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right) + \sum_{n=-\infty}^{-1} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right)$$

Define new coefficients

$$U_0 \equiv a_0 = \frac{1}{T}\int_0^T \hat{u}(t)\,\mathrm{dt} \equiv \frac{1}{T}\int_0^T \hat{u}(t)\,e^{i\,(0)\,\omega\,t}\,\mathrm{dt}$$

$$U_n = \frac{a_n}{2} + \frac{b_n}{2\,i} = \frac{1}{2\,T}\int_0^T \hat{u}(t)\left(e^{i\,n\,\omega\,t} + e^{-i\,n\,\omega\,t}\right)\mathrm{dt} - \frac{1}{2\,T}\int_0^T \hat{u}(t)\left(e^{i\,n\,\omega\,t} - e^{-i\,n\,\omega\,t}\right)\mathrm{dt}$$

$$\Longrightarrow U_n = \frac{1}{T}\int_0^T \hat{u}(t)\,e^{-i\,n\,\omega\,t}\,\mathrm{dt}; \quad n = \pm 1,\ \pm 2,\ \ldots$$

$$U_n = \frac{1}{T}\int_0^T \hat{u}(t)\,e^{-i\,n\,\omega\,t}\,\mathrm{dt}; \quad n = 0,\ 1,\ 2,\ \ldots\,; \quad U_{-n} = \mathrm{Conj}(U_n)$$

$$u(t) = a_0 + \sum_{n=1}^{\infty} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right) + \sum_{n=-\infty}^{-1} e^{i\,n\,\omega\,t}\left(\frac{a_n}{2} + \frac{b_n}{2\,i}\right)$$

$$u(t) = U_0\,e^{i\,(0)\,\omega\,t} + \sum_{n=1}^{\infty} U_n\,e^{i\,(n\,\omega\,t)} + \sum_{n=-\infty}^{-1} U_{-n}\,e^{i\,(-n\,\omega\,t)}$$

or

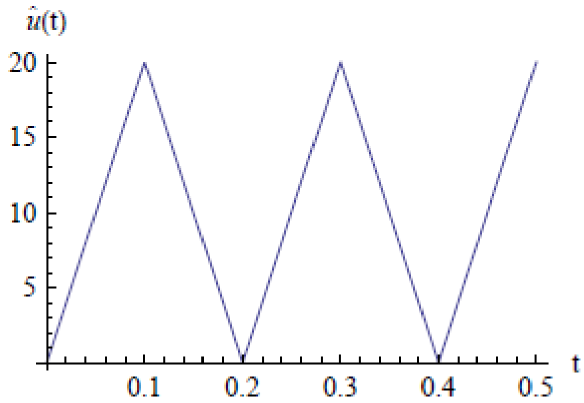
$$u(t) = \sum_{n=-\infty}^{\infty} U_n\,e^{i\,n\,\omega\,t}$$

In summary,

$$U_n = \frac{1}{T}\int_0^T \hat{u}(t)\,e^{-i\,n\,\omega\,t}\,\mathrm{dt}; \quad n = 0,\ 1,\ 2,\ \ldots\,; \quad U_{-n} = \mathrm{Conj}(U_n)$$

$$u(t) = \sum_{n=-\infty}^{\infty} U_n\,e^{i\,n\,\omega\,t}$$ or $$u(t) = \sum_{n=-p}^{p} U_n\,e^{i\,n\,\omega\,t}$$

Example: Compute and draw frequency spectrum for the following function



Period, T = 0.2; $\omega = 31.4159$

$$\hat{u}(t) = \begin{cases} 200\,t & \text{if} \quad t \le 0.1 \\ 40 - 200\,t & \text{if} \quad t > 0.1 \end{cases}$$

$$\hat{u}(t) = \text{If}\left[t \le 0.1, \frac{40\,t}{0.2}, \frac{40\,(0.2 - t)}{0.2}\right]; \qquad \text{Period, T = 0.2;} \qquad \omega = 31.4159$$

$$U_0 = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(0)(31.4159)t}\, dt = 10.$$

$$U_1 = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(1)(31.4159)t}\, dt = -4.05285$$

$$U_2 = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(2)(31.4159)t}\, dt = 0$$

$$U_3 = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(3)(31.4159)t}\, dt = -0.450316$$

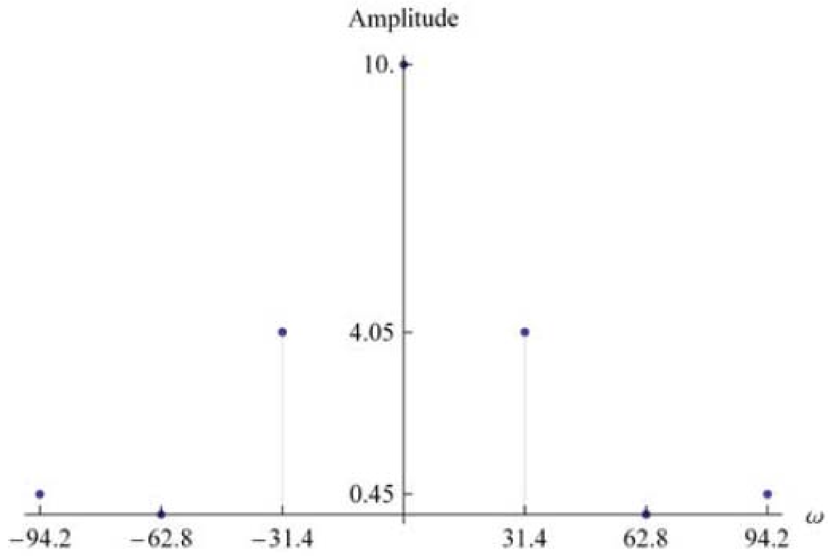
$$U_{-3} = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(-3)(31.4159)t}\, dt = -0.450316$$

$$U_{-2} = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(-2)(31.4159)t}\, dt = 0$$

$$U_{-1} = (1/T)\int_0^{0.2} \hat{u}\, e^{(-i)(-1)(31.4159)t}\, dt = -4.05285$$

$$u(t) = 10. - 4.05285\, e^{-31.4159\, i\, t} - 4.05285\, e^{31.4159\, i\, t} - 0.450316\, e^{-94.2478\, i\, t} - 0.450316\, e^{94.2478\, i\, t}$$

	1	2	3	4	5	6	7
Frequency	−94.2478	−62.8319	−31.4159	0	31.4159	62.8319	94.2478
Amplitude	0.450316	0	4.05285	10.	4.05285	0	0.450316



$$u(t) = 10. - 4.05285\, e^{-31.4159\, i\, t} - 4.05285\, e^{31.4159\, i\, t} - 0.450316\, e^{-94.2478\, i\, t} - 0.450316\, e^{94.2478\, i\, t}$$