Lecture 10: Multiple degrees of freedom system using modal superposition (Optional)

Reading materials: Section 3.8

1. Equations of motion

$$\boldsymbol{m}\,\ddot{\boldsymbol{u}} + \boldsymbol{c}\,\dot{\boldsymbol{u}} + \boldsymbol{k}\,\boldsymbol{u} = \boldsymbol{f}(t); \quad \boldsymbol{u}(0) = \boldsymbol{u}_0; \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0$$

$$\boldsymbol{f}(t) = \boldsymbol{F}\sin\left(\omega_f\, t\right)$$

- Rayleigh damping

$$\boldsymbol{c} = \alpha\,\boldsymbol{m} + \beta\,\boldsymbol{k}$$

2. Uncoupled equations

- Undamped free vibration mode shapes and frequencies

$$\boldsymbol{k}\,\boldsymbol{\phi}_i = \lambda_i\,\boldsymbol{m}\,\boldsymbol{\phi}_i; \quad i = 1, 2, \ldots, n$$

$$\boldsymbol{\phi}_j^T\,\boldsymbol{m}\,\boldsymbol{\phi}_i = 0; \quad i \neq j \qquad \boldsymbol{\phi}_j^T\,\boldsymbol{k}\,\boldsymbol{\phi}_i = 0; \quad i \neq j$$

- Uncoupled equations

$$\boldsymbol{z} = \begin{pmatrix} z_1 & z_2 & \ldots & z_n \end{pmatrix}^T$$

$$\boldsymbol{u}(t) = \sum_i z_i(t)\,\boldsymbol{\phi}_i$$

$$M_i\,\ddot{z}_i(t) + (\alpha\,M_i + \beta\,K_i\,)\,\dot{z}_i(t) + K_i\,z_i(t) = R_i \sin\left(\omega_f\,t\right);\quad i = 1, 2, \ldots$$

$$z_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\,\boldsymbol{m}\,\boldsymbol{u}^0\right);\quad \dot{z}_i(0) = \frac{1}{M_i}\left(\boldsymbol{\phi}_i^T\,\boldsymbol{m}\,\boldsymbol{v}^0\right);$$

$$M_i = \boldsymbol{\phi}_i^T\,\boldsymbol{m}\,\boldsymbol{\phi}_i;\quad K_i = \boldsymbol{\phi}_i^T\,\boldsymbol{k}\,\boldsymbol{\phi}_i;\quad \omega_i = \sqrt{K_i\,/\,M_i}\,;\quad R_i = \boldsymbol{\phi}_i^T\,\boldsymbol{F}$$

$$C_i = \alpha\,M_i + \beta\,K_i \equiv 2\,M_i\,\xi_i\,\omega_i$$

$$M_i\,\ddot{z}_i(t) + C_i\,\dot{z}_i(t) + K_i\,z_i(t) = R_i \sin\left(\omega_f\,t\right);\quad i = 1, 2, \ldots$$

Solution

$$\boldsymbol{u}\,(t) = \textstyle\sum_i z_i(t)\,\boldsymbol{\phi}_i$$

$$z_i(t) = e^{-\xi_i\,\omega_i\,t}(A_i \cos\omega_{\text{di}}\,t + B_i \sin\omega_{\text{di}}\,t)\ + \frac{R_i/K_i}{\gamma_i}\left[\left(1 - r_i^2\right)\sin\,\omega_f\,t - 2\,\xi_i\,r_i \cos\,\omega_f\,t\right]$$

$$r_i = \frac{\omega_f}{\omega_i}$$

$$\gamma_i = \left(1 - r_i^2\right)^2 + \ (2\,\xi_i\,r_i)^2$$

$$A_i = u_0 + \frac{2\,R_i\,r_i\,\xi_i}{K_i\,\gamma_i};$$

$$B_i = \frac{\xi_i\,\omega_i\,u_0}{\omega_{\text{di}}} + \frac{v_0}{\omega_{\text{di}}} - \frac{R_i\left(\left(1-r_i^2\right)\omega_f - 2\,r_i\,\omega_i\,\xi_i^2\right)}{K_i\,\gamma_i\,\omega_{\text{di}}}$$