A Spline Chaos Expansion for Uncertainty Quantification in Linear Dynamical Systems

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Outline

UQ in Frequency Response Analysis

- A linear, M-DOF, stochastic dynamic system satisfies

$$\mathbf{M}(\mathbf{X})\ddot{\mathbf{z}}(t;\mathbf{X}) + \mathbf{C}(\mathbf{X})\dot{\mathbf{z}}(t;\mathbf{X}) + \mathbf{K}(\mathbf{X})\mathbf{z}(t;\mathbf{X}) = \mathbf{f}(t).$$

- For $\mathbf{f}(t) = \mathbf{F}(\omega)\exp(\mathrm{i}\omega t)$, the steady-state displ. response is $\mathbf{z}(t) = \mathbf{Z}(\omega;\mathbf{X})\exp(\mathrm{i}\omega t)$, where the displ. ampl. $\mathbf{Z}(\omega;\mathbf{X})$ satisfies

$$\left[-\omega^2\mathbf{M}(\mathbf{X}) + \mathrm{i}\omega\mathbf{C}(\mathbf{X}) + \mathbf{K}(\mathbf{X})\right]\mathbf{Z}(\omega;\mathbf{X}) = \mathbf{F}(\omega),$$

$$\mathbf{Z}(\omega;\mathbf{X}) = \underbrace{\left[-\omega^2\mathbf{M}(\mathbf{X}) + \mathrm{i}\omega\mathbf{C}(\mathbf{X}) + \mathbf{K}(\mathbf{X})\right]^{-1}}_{:=\mathbf{H}(\omega;\mathbf{X})}\mathbf{F}(\omega) = \underbrace{\mathbf{H}(\omega;\mathbf{X})}_{\text{FRF}}\mathbf{F}(\omega).$$

- $\mathbf{X} = (X_1,\ldots,X_N)^\intercal \rightarrow N$-dim. input random vector representing uncertainties in mass, damping, and stiffness matrices.
- Given the probability law of $\mathbf{X}$, what are the statistical properties (mean, variance, *etc.*) of random FRFs or displ. amplitudes?

UQ Challenges & Methods

$$\text{Input } \mathbf{X} = (X_1, \ldots, X_N) \rightarrow \boxed{\textbf{DYNAMIC SYSTEM}} \rightarrow \text{Output } Y = y(\mathbf{X})$$

$$Y = g(\mathbf{Z}(\omega; \mathbf{X})) =: y(\mathbf{X}) \quad \text{(Frequency Response Analysis)}$$

- **Challenges (Works at Iowa)**
  - Locally prominent (nonsmoothness, discontinuity) responses
  - High-dimensional random input ($N \geq 10$)
- **Popular (Existing) Methods**
  - PCE, PDD, stochastic collocation, sparse grids, and others
  - Most methods break down for non-smooth/discontinuous responses

Explore orthogonal splines with local support

Assumptions

The random vector $\mathbf{X} := (X_1, \ldots, X_N)^\intercal : (\Omega, \mathcal{F}) \to (\mathbb{A}^N, \mathcal{B}^N)$ satisfies the following conditions:

1. All component random variables X_k, $k = 1, \ldots, N$, are statistically independent, but not necessarily identical.
2. Each input random variable X_k has absolute continuous marginal CDF and continuous marginal PDF.
3. Each input random variable X_k is defined on a closed bounded interval $[a_k, b_k] \subset \mathbb{R}$, $b_k > a_k$, so that all moments exist, *i.e.*, for $l \in \mathbb{N}_0$,

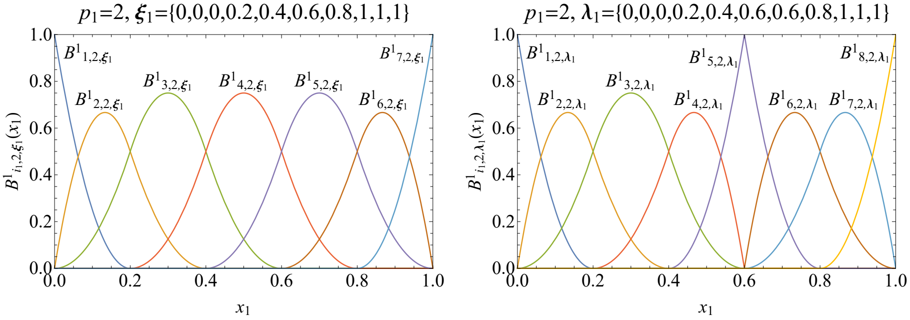
$$\mathbb{E}\left[X_k^l\right] := \int_\Omega X_k^l(\omega)\, d\mathbb{P}(\omega) = \int_{a_k}^{b_k} x_k^l f_{X_k}(x_k)\, dx_k < \infty.$$

Univariate B-Splines (Cox & de Boor, 1972)

For a knot sequence $\boldsymbol{\xi}_k = \{a_k = \xi_{k,1}, \ldots, \xi_{k,n_k+p_k+1} = b_k\}$, where $\xi_{k,1} \leq \cdots \leq \xi_{k,n_k+p_k+1}$, $n_k > p_k \geq 0$, the B-splines are

$$B^k_{i_k,p_k,\boldsymbol{\xi}_k}(x_k) := \frac{(x_k - \xi_{k,i_k})B^k_{i_k,p_k-1,\boldsymbol{\xi}_k}(x_k)}{\xi_{k,i_k+p_k} - \xi_{k,i_k}} + \frac{(\xi_{k,i_k+p_k+1} - x_k)B^k_{i_k+1,p_k-1,\boldsymbol{\xi}_k}(x_k)}{\xi_{k,i_k+p_k+1} - \xi_{k,i_k+1}},$$

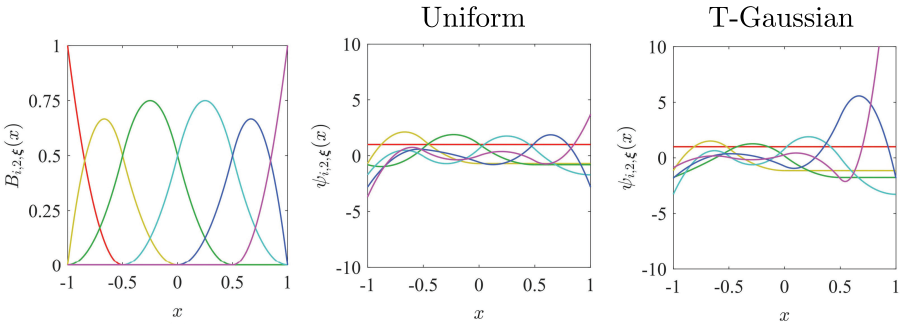
$$1 \leq k \leq N, 1 \leq i_k \leq n_k, 1 \leq p_k < \infty.$$



Orthonormalized Univariate B-Splines

For $k = 1, \ldots, N$, let $B^k_{i_k, p_k, \boldsymbol{\xi}_k}(x_k)$ & $\psi^k_{i_k, p_k, \boldsymbol{\xi}_k}(x_k)$ be real-valued B-splines and ON B-splines in x_k of degree $p_k \in \mathbb{N}_0$ and knot sequence $\boldsymbol{\xi}_k = \{a_k = \xi_{k,1}, \ldots, \xi_{k, n_k + p_k + 1} = b_k\}$, $n_k > p_k \geq 0$.

Example: $p_k = 2$, $\boldsymbol{\xi}_k = \{-1, -1, -1, -0.5, 0, 0.5, 1, 1, 1\}$.



Multivariate ON B-Splines (Full Tensor-Product)

For $\mathbf{i} := (i_1, \ldots, i_N)$, $\mathbf{p} := (p_1, \ldots, p_N)$, $\boldsymbol{\Xi} := (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N)$, the tensor-product ON B-splines in $\mathbf{x} = (x_1, \ldots, x_N)$ are

$$\Psi_{\mathbf{i},\mathbf{p},\boldsymbol{\Xi}}(\mathbf{x}) = \prod_{k=1}^{N} \psi^k_{i_k,p_k,\boldsymbol{\xi}_k}(x_k), \; \mathcal{S}_{\mathbf{p},\boldsymbol{\Xi}} = \operatorname{span}\left\{\Psi_{\mathbf{i},\mathbf{p},\boldsymbol{\Xi}}(\mathbf{x})\right\}_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}.$$

$$\mathcal{I}_{\mathbf{n}} := \{\mathbf{i} = (i_1, \ldots, i_N) : 1 \le i_k \le n_k, k = 1, \ldots, N\}$$

The second-moment properties are

$$\mathbb{E}\left[\Psi_{\mathbf{i},\mathbf{p},\boldsymbol{\Xi}}(\mathbf{X})\right] = \begin{cases} 1, & \mathbf{i} = \mathbf{1} := (1, \ldots, 1), \\ 0, & \mathbf{i} \neq \mathbf{1}. \end{cases}$$

$$\mathbb{E}\left[\Psi_{\mathbf{i},\mathbf{p},\boldsymbol{\Xi}}(\mathbf{X}_u)\Psi_{\mathbf{j},\mathbf{p},\boldsymbol{\Xi}}(\mathbf{X}_v)\right] = \begin{cases} 1, & \mathbf{i} = \mathbf{j}, \\ 0, & \mathbf{i} \neq \mathbf{j}. \end{cases}$$

Spline Chaos Expansion (SCE)

Theorem

Under Assumptions 1-3, a random variable $y(\mathbf{X}) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ *admits an orthogonal expansion in multivariate ON spline basis* $\{\Psi_{\mathbf{i},\mathbf{p},\mathbf{\Xi}}(\mathbf{X})\}$, *referred to as the SCE of*

$$y_{\mathbf{p},\mathbf{\Xi}}(\mathbf{X}) := \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} C_{\mathbf{i},\mathbf{p},\mathbf{\Xi}} \Psi_{\mathbf{i},\mathbf{p},\mathbf{\Xi}}(\mathbf{X}),$$

where

$$C_{\mathbf{i},\mathbf{p},\mathbf{\Xi}} := \int_{\mathbb{A}^N} y(\mathbf{x}) \Psi_{\mathbf{i},\mathbf{p},\mathbf{\Xi}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}.$$

$$\mathbb{E}\left[\left|y(\mathbf{X}) - y_{\mathbf{p},\mathbf{\Xi}}(\mathbf{X})\right|^2\right] \leq C\omega_{\mathbf{p}+\mathbf{1}}(y;\mathbf{h})_{L^2(\mathbb{A}^N)}$$

$$\lim_{\mathbf{h}\to\mathbf{0}} \mathbb{E}\left[\left|y(\mathbf{X}) - y_{\mathbf{p},\mathbf{\Xi}}(\mathbf{X})\right|^2\right] = 0$$

Output Statistics

- **Mean and Variance**

$$\mathbb{E}\left[y_{\mathbf{p},\mathbf{\Xi}}(\mathbf{X})\right] = C_{\mathbf{1},\mathbf{p},\mathbf{\Xi}} = \mathbb{E}\left[y(\mathbf{X})\right]$$

$$\mathrm{var}\left[y_{\mathbf{p},\mathbf{\Xi}}(\mathbf{X})\right] = \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} C^2_{\mathbf{i},\mathbf{p},\mathbf{\Xi}} - C^2_{\mathbf{1},\mathbf{p},\mathbf{\Xi}} \leq \mathrm{var}\left[y(\mathbf{X})\right]$$

- **No. of Basis Functions**

$$L_{\mathbf{p},\mathbf{\Xi}} = |\mathcal{I}_{\mathbf{n}}| = \prod_{k=1}^{N} n_k$$

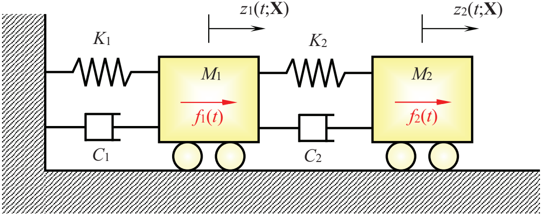
SCE suffers from the curse of dimensionality.

A 2-DOF System with Random Spring Constants

$$M_1 = M_2 = 1 \text{ kg}, \quad C_1 = C_2 = 1 \text{ N/(ms)},$$

$$K_1 = K_2 = 15000(1 + 0.05 X_K) \text{ N/m}, \quad X_K \sim N(0,1)$$

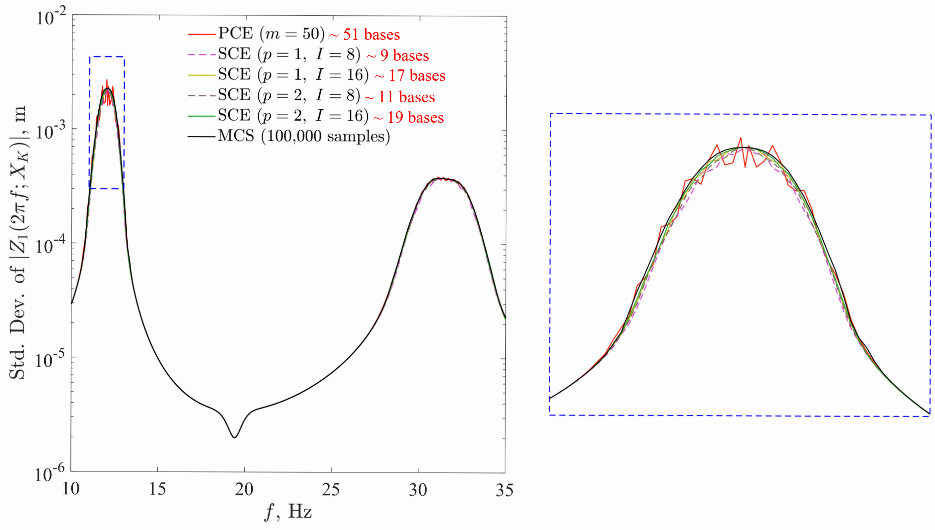
Natural Freq. at Mean Input: 12.05 Hz; 31.54 Hz



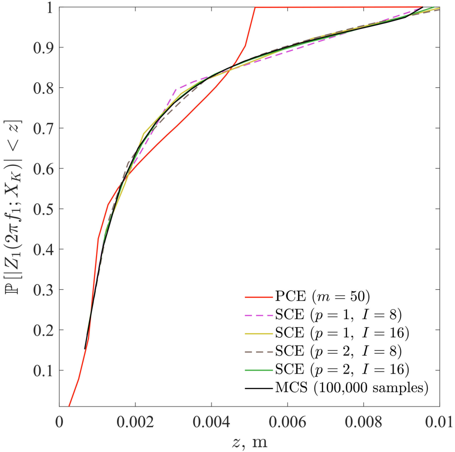
$$\left(-\omega^2 \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} + \mathrm{i}\omega \begin{bmatrix} C_1 + C_2 & -C_2 \\ -C_2 & C_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\right) \begin{pmatrix} Z_1(\omega; X_K) \\ Z_2(\omega; X_K) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

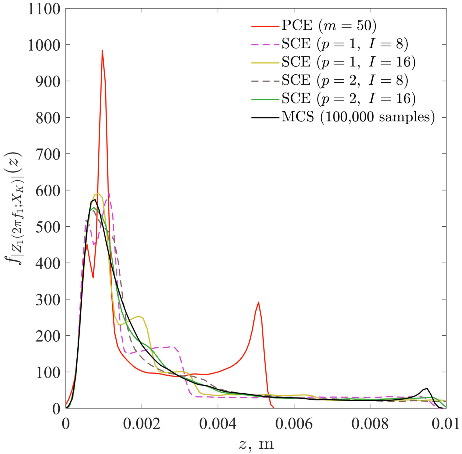
What are the probabilistic characteristics of $Z_i(\omega; X_K)$, $i = 1, 2$?

St. Dev. of FRF


PCE ($m = 50$) ~ 51 bases
SCE ($p = 1$, $I = 8$) ~ 9 bases
SCE ($p = 1$, $I = 16$) ~ 17 bases
SCE ($p = 2$, $I = 8$) ~ 11 bases
SCE ($p = 2$, $I = 16$) ~ 19 bases
MCS (100,000 samples)
Std. Dev. of $|Z_1(2\pi f; X_K)|$, m
f, Hz
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10
15
20
25
30
35

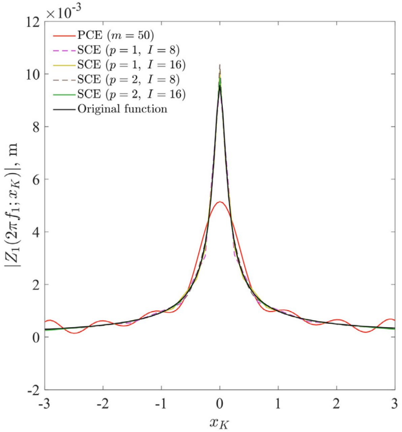
PDF & CDF of FRF





Non-Smooth Function

(b) Non-uniformly spaced knots	
1st-order ($p = 1,\ I = 8$)	$\xi = \{-3, -3, -1.5, -0.65, -0.25, 0, 0.25, 0.65, 1.5, 3, 3\}$
1st-order ($p = 1,\ I = 16$)	$\xi = \{-3, -3, -2.25, -1.75, -1.25, -0.8, -0.4, -0.25, -0.1, 0, 0.1, 0.25, 0.4, 0.8, 1.25, 1.75, 2.25, 3, 3\}$
2nd-order ($p = 2,\ I = 8$)	$\xi = \{-3, -3, -3, -1.5, -0.65, -0.25, 0, 0, 0.25, 0.65, 1.5, 3, 3, 3\}$
2nd-order ($p = 2,\ I = 16$)	$\xi = \{-3, -3, -3, -2.25, -1.75, -1.25, -0.8, -0.4, -0.25, -0.1, 0, 0, 0.1, 0.25, 0.4, 0.8, 1.25, 1.75, 2.25, 3, 3, 3\}$



Conclusion

- A novel SCE for probabilistic freq. response analysis of dynamic systems
- SCE tackles non-smooth functions better than PCE
- SCE provides more accurate estimates of output statistics and PDF/CDF than PCE
- SCE suffers from the curse of dimensionality

Reference

Rahman, S. and Jahanbin, R., "Orthogonal Spline Expansions for Uncertainty Quantification in Linear Dynamical Systems," *Journal of Sound and Vibration*, Vol. 512, Article 116366, pp. 1-25, 2021.