Higher-Order Moments of Spline Chaos Expansion

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Outline

Uncertainty Quantification

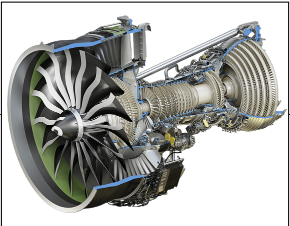
Complex System

(jet engine)

Input $\mathbf{X} = (X_1, \ldots, X_N)$

$\mathbf{X} : (\Omega, \mathcal{F}) \to (\mathbb{A}^N, \mathcal{B}^N) \to$

$\mathbb{A}^N \subseteq \mathbb{R}^N, \;\; N \in \mathbb{N}$



$\to$ Output $Y = y(\mathbf{X})$

$Y \in L^p(\Omega, \mathcal{F}, \mathbb{P})$

$y \in L^p(\mathbb{A}^N, \mathcal{B}^N, f_{\mathbf{X}}\, d\mathbf{x})$

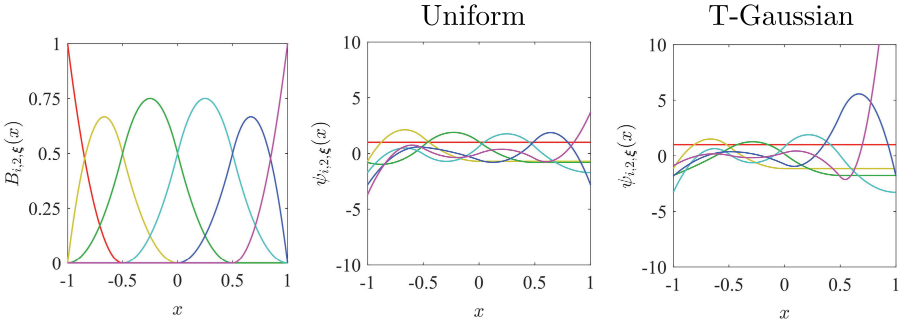
- **Goals & Objectives**
  - Moments: $\mathbb{E}\left[Y^l\right] := \int_\Omega Y^l \, d\mathbb{P} = \int_{\mathbb{A}^N} y^l(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$,
  - Probability distribution: $\mathbb{P}\left[Y \leq y_0\right] := \int_{\{\mathbf{x}: y(\mathbf{x}) \leq y_0\}} f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$
  - Stochastic design optimization (RDO/RBDO)

Orthonormalized (ON) Univariate B-Splines

For $k = 1, \ldots, N$, let $B^k_{i_k,m_k,\boldsymbol{\xi}_k}(x_k)$ & $\psi^k_{i_k,m_k,\boldsymbol{\xi}_k}(x_k)$ be real-valued B-splines and ON B-splines in x_k of degree $m_k \in \mathbb{N}_0$ and knot sequence

$$\boldsymbol{\xi}_k = \{a_k = \xi_{k,1}, \ldots, \xi_{k,n_k+m_k+1} = b_k\},\ n_k = \mathrm{len}(\boldsymbol{\xi}_k) - m_k - 1.$$

Example: $m_k = 2$, $\boldsymbol{\xi}_k = \{-1, -1, -1, -0.5, 0, 0.5, 1, 1, 1\}$, $n_k = 6$.



Multivariate ON B-Splines (Full Tensor-Product)

For $\mathbf{i} := (i_1, \ldots, i_N)$, $\mathbf{m} := (m_1, \ldots, m_N)$, $\boldsymbol{\Xi} := (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N)$, the tensor-product ON B-splines in $\mathbf{x} = (x_1, \ldots, x_N)$ are

$$\boxed{\Psi_{\mathbf{i},\mathbf{m},\boldsymbol{\Xi}}(\mathbf{x}) = \prod_{k=1}^{N} \psi^k_{i_k, m_k, \boldsymbol{\xi}_k}(x_k), \ \mathcal{S}_{\mathbf{m},\boldsymbol{\Xi}} = \operatorname{span}\left\{\Psi_{\mathbf{i},\mathbf{m},\boldsymbol{\Xi}}(\mathbf{x})\right\}_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}.}$$

$$\mathcal{I}_{\mathbf{n}} := \{\mathbf{i} = (i_1, \ldots, i_N) : 1 \le i_k \le n_k, k = 1, \ldots, N\}$$

The second-moment properties are

$$\boxed{\begin{aligned} \mathbb{E}\left[\Psi_{\mathbf{i},\mathbf{m},\boldsymbol{\Xi}}(\mathbf{X})\right] &= \begin{cases} 1, & \mathbf{i} = \mathbf{1} := (1, \ldots, 1), \\ 0, & \mathbf{i} \neq \mathbf{1}. \end{cases} \\ \mathbb{E}\left[\Psi_{\mathbf{i},\mathbf{m},\boldsymbol{\Xi}}(\mathbf{X}_u)\Psi_{\mathbf{j},\mathbf{m},\boldsymbol{\Xi}}(\mathbf{X}_v)\right] &= \begin{cases} 1, & \mathbf{i} = \mathbf{j}, \\ 0, & \mathbf{i} \neq \mathbf{j}. \end{cases} \end{aligned}}$$

Spline Chaos Expansion (SCE) [Rahman, 2020]

Theorem

For independent random input $\mathbf{X}$, *an output random variable* $y(\mathbf{X}) \in L^p(\Omega, \mathcal{F}, \mathbb{P})$, $2 \le p < \infty$, *admits an orthogonal expansion in* $\{\Psi_{\mathbf{i},\mathbf{m},\mathbf{\Xi}}(\mathbf{X})\}_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}$, *referred to as the SCE of*

$$y_{\mathbf{m},\mathbf{\Xi}}(\mathbf{X}) := \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} C_{\mathbf{i},\mathbf{m},\mathbf{\Xi}} \Psi_{\mathbf{i},\mathbf{m},\mathbf{\Xi}}(\mathbf{X}),$$

where

$$C_{\mathbf{i},\mathbf{m},\mathbf{\Xi}} := \int_{\mathbb{A}^N} y(\mathbf{x}) \Psi_{\mathbf{i},\mathbf{m},\mathbf{\Xi}}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}.$$

$$\boxed{L_{\mathbf{m},\mathbf{\Xi}} = |\mathcal{I}_{\mathbf{n}}| = \prod_{k=1}^{N} n_k}$$

SCE suffers from the curse of dimensionality

Error Bound & L^p Convergence ($2 \le p < \infty$)

- **α_kth modulus of smoothness of y in L^p** ($\alpha_k \ge 1$)

$$\omega_{\alpha_k}(y; h_k)_{L^p[a_k,b_k]} := \sup_{0 \le u_k \le h_k} \left\| \Delta_{u_k}^{\alpha_k} y(x_k) \right\|_{L^p[a_k, b_k - \alpha_k u_k]}, \quad h_k \ge 0,$$

$$\Delta_{u_k}^{\alpha_k} y(x_k) := \sum_{i=0}^{\alpha_k} (-1)^{\alpha_k - i} \binom{\alpha_k}{i} y(x_k + i u_k) \to \alpha_k \text{th fwd. diff.}$$

$$\omega_{\boldsymbol{\alpha}}(y; \mathbf{h})_{L^p[\mathbb{A}^N]} := \sup_{\mathbf{0} \le \mathbf{u} \le \mathbf{h}} \left\| \Delta_{\mathbf{u}}^{\boldsymbol{\alpha}} y(\mathbf{x}) \right\|_{L^p[\mathbb{A}^N_{\boldsymbol{\alpha},\mathbf{u}}]}, \quad \mathbf{h} \ge \mathbf{0}$$

- **L^p-error**

$$\boxed{\begin{gathered} \mathbb{E}\left[|y(\mathbf{X}) - y_{\mathbf{m},\boldsymbol{\Xi}}(\mathbf{X})|^p \right] \le C \omega_{\mathbf{m+1}}(y; \mathbf{h})_{L^p(\mathbb{A}^N)} \\ \lim_{\mathbf{h} \to \mathbf{0}} \mathbb{E}\left[|y(\mathbf{X}) - y_{\mathbf{m},\boldsymbol{\Xi}}(\mathbf{X})|^p \right] = 0 \end{gathered}}$$

SCE converges in the pth mean, in prob., & in distribution

Output Statistics

- **Mean and Variance**

$$\mu_{\mathbf{m},\mathbf{\Xi}} := \mathbb{E}\left[y_{\mathbf{m},\mathbf{\Xi}}(\mathbf{X})\right] = C_{\mathbf{1},\mathbf{m},\mathbf{\Xi}} = \mathbb{E}\left[y(\mathbf{X})\right]$$

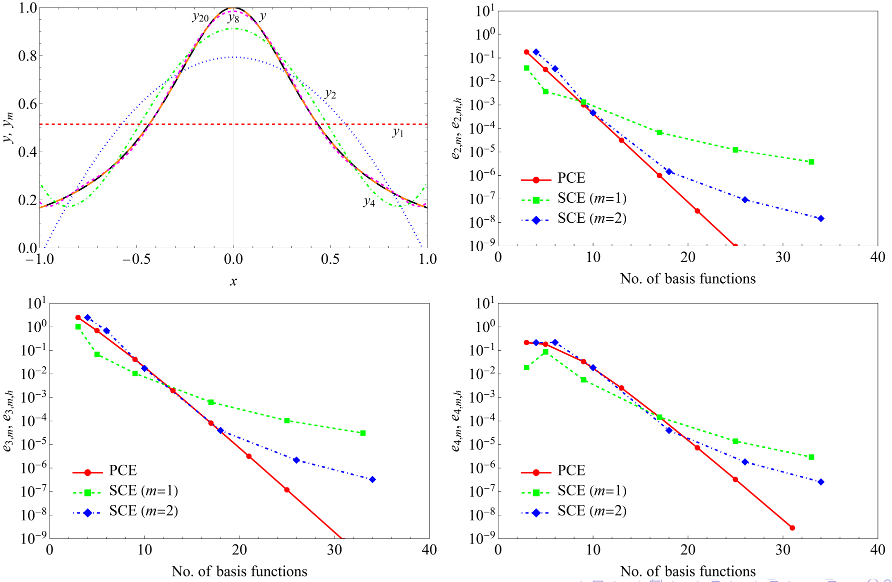
$$\sigma^2_{\mathbf{m},\mathbf{\Xi}} := \mathrm{var}\left[y_{\mathbf{m},\mathbf{\Xi}}(\mathbf{X})\right] = \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} C^2_{\mathbf{i},\mathbf{m},\mathbf{\Xi}} - C^2_{\mathbf{1},\mathbf{m},\mathbf{\Xi}} \leq \mathrm{var}\left[y(\mathbf{X})\right]$$

- **Skewness and Kurtosis**

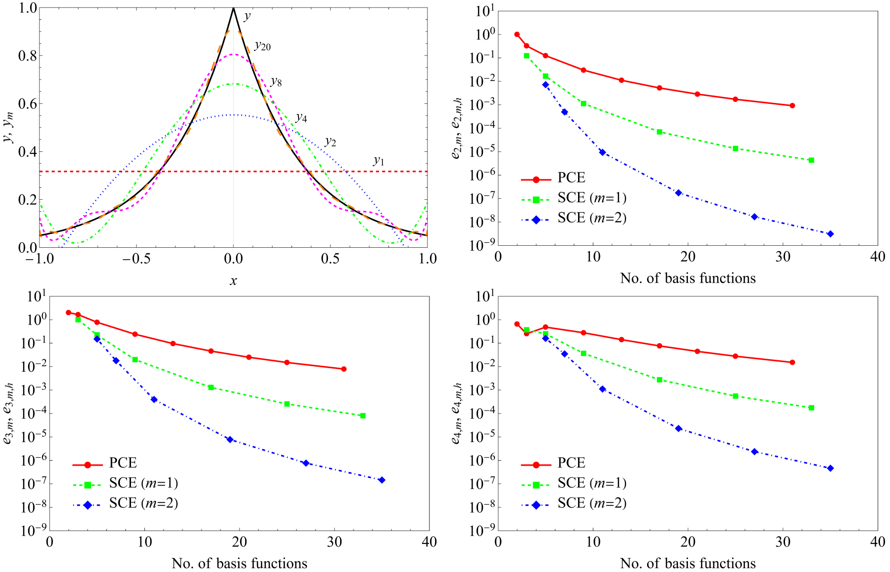
$$\gamma_{\mathbf{m},\mathbf{\Xi}} := \mathbb{E}\left[\frac{y_{\mathbf{m},\mathbf{\Xi}}(\mathbf{X}) - \mu_{\mathbf{m},\mathbf{\Xi}}}{\sigma_{\mathbf{m},\mathbf{\Xi}}}\right]^3$$

$$\kappa_{\mathbf{m},\mathbf{\Xi}} := \mathbb{E}\left[\frac{y_{\mathbf{m},\mathbf{\Xi}}(\mathbf{X}) - \mu_{\mathbf{m},\mathbf{\Xi}}}{\sigma_{\mathbf{m},\mathbf{\Xi}}}\right]^4$$

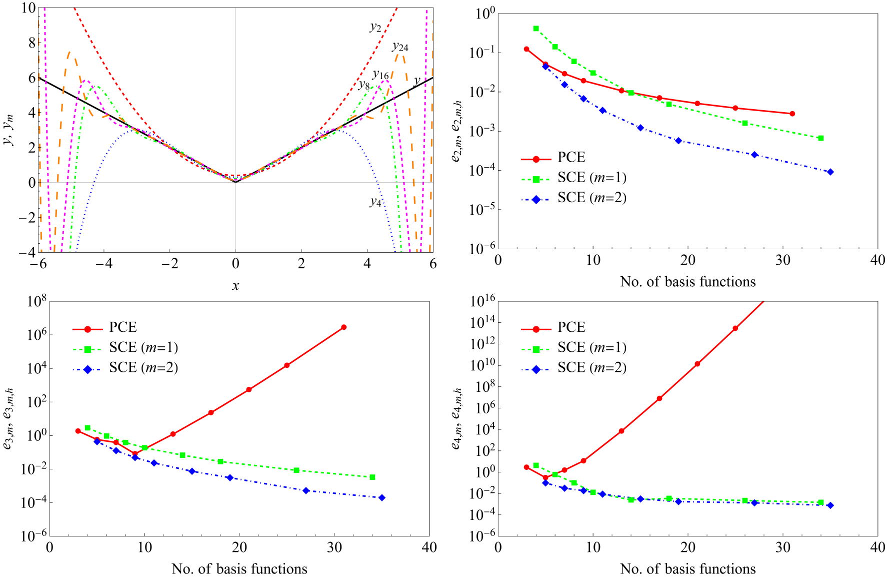
1. $y(X) = 1/(1+5X^2)$, $X \sim U[-1,+1]$



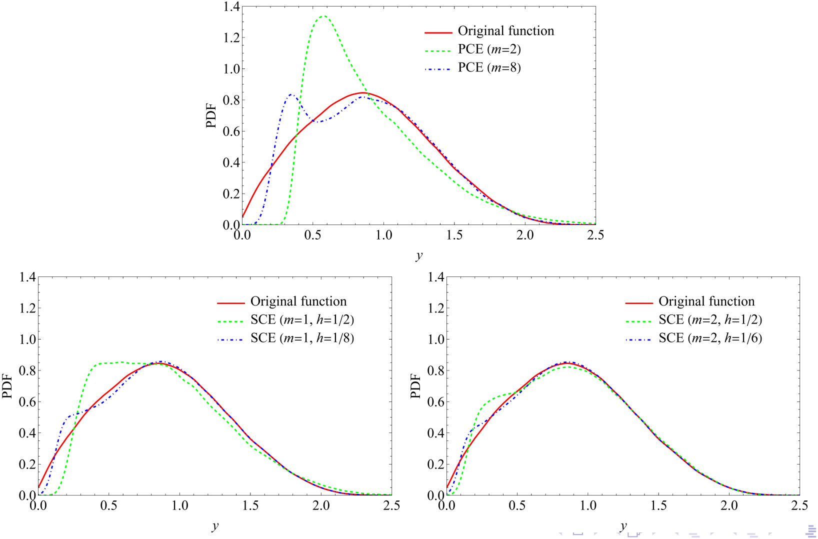
2. $y(X) = \exp(-3|X|)$, $X \sim U[-1, +1]$



3. $y(X) = |X|,\ X \sim N[0,1]$



4. $y(\mathbf{X}) = \prod_{i=1}^{4} \frac{|4X_i-2|^{b_i}+a_i}{1+a_i}, \; X_1, \cdots, X_4 \overset{i.i.d.}{\sim} U[0,1]$



Conclusion

- Proof that SCE converges in L^p
- SCE more efficient than PCE for non-smooth functions
- Higher-order moments of PCE may diverge
- SCE and PCE suffer from the curse of dimensionality

Reference

Rahman, S., "Higher-Order Moments of Spline Chaos Expansion," *Probabilistic Engineering Mechanics*, Vol. 77, Article 103666, pp. 1-15, 2024.