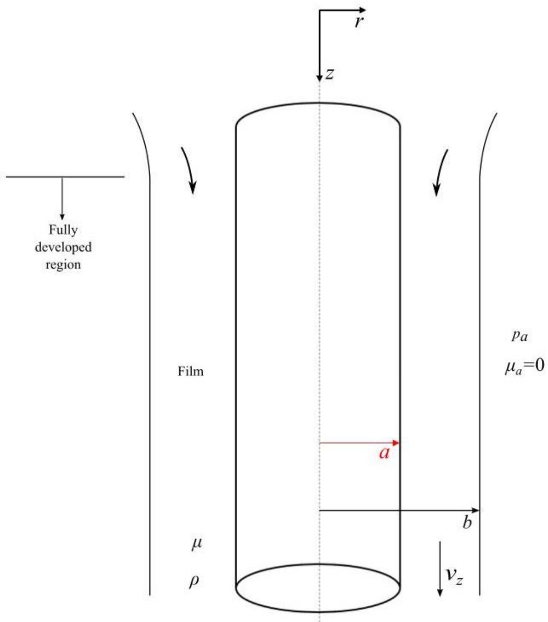
Name: ------------------ **Quiz 7** **Time: 20 minutes**

ME:5160 **Fall 2024**

The exam is closed book and closed notes.

Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius a, as shown in the Figure. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius b, with $v_z = v_z(r), v_\theta = v_r = 0$. Assume that the atmosphere offers no shear resistance to the film motion and that $\frac{\partial p}{\partial z} = 0$. (a) Using continuity, how that v_z is independent from z. (b) Derive a differential equation for v_z, stating and applying the proper boundary conditions and solve for the film velocity distribution. (c) Determine the value of the shear stress at r=a.



Boundary condition Hint

- At the pipe wall, the velocity is zero
- At the film/air interface, the shear stress is zero ⟶ $\mu \frac{\partial v_z}{\partial r} = 0$

The equations of motion of an incompressible Newtonian fluid with constant density and viscosity in cylindrical coordinates (r, θ, z) with velocity components (v_r, v_θ, v_z):

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

r-momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]$$

θ-momentum:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

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Solution:

ASSUMPTIONS:

1. Steady flow ($\frac{\partial}{\partial t}$=0)
2. Incompressible flow (ρ=constant)
3. Purely axial flow (vr=vθ=0)
4. Circumferentially symmetric flow, so properties do not vary with θ ($\frac{\partial}{\partial \theta}$=0)
5. Zero pressure gradient ($\partial p/\partial$z =0)
6. Vertical motion (gz=g)

(a)

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta)+\frac{\partial}{\partial z}(v_z)=0$$

$$0(3)+0(3)+\frac{\partial v_z}{\partial z}=0 \qquad \textbf{(+1.5)}$$

z-momentum:

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

$$\rho(0(1)+0(3)+0(3{,}4)+0(continuity))=\rho g-0(5)+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+0(4)+0(continuity)\right]$$

$$\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=-\rho g$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=-\frac{\rho g}{\mu}r$$

$$r\frac{\partial v_z}{\partial r}=-\frac{\rho g}{2\mu}r^2+c_1$$

$$\frac{\partial v_z}{\partial r}=-\frac{\rho g}{2\mu}r+\frac{c_1}{r}$$

$$v_z(r)=-\frac{\rho g}{4\mu}r^2+c_1\ln r+c_2$$

Boundary conditions:

$$v_z(a)=0, \qquad \mu\frac{\partial v_z}{\partial r}(b)=0$$

Apply BCs:

$$v_z(a) = 0 = -\frac{\rho g}{4\mu}a^2 + c_1 \ln a + c_2$$

$$\frac{\partial v_z}{\partial r} = -\frac{\rho g}{2\mu} r + \frac{c_1}{r}$$

$$\frac{\partial v_z}{\partial r}(b) = 0 = -\frac{\rho g}{2\mu} b + \frac{c_1}{b}$$

$$c_1 = \frac{\rho g}{2\mu} b^2$$

$$c_2 = \frac{\rho g}{4\mu} a^2 - \frac{\rho g}{2\mu} b^2 \ln a$$

Film velocity distribution:

$$v_z(r) = -\frac{\rho g}{4\mu}(r^2 - a^2) + \frac{\rho g b^2}{2\mu} \ln(r/a)$$

Shear stress:

$$\tau_w(r) = \mu \frac{\partial v_z}{\partial r} = -\frac{\rho g r}{2\mu} + \frac{\rho g b^2}{2\mu r}$$

$$\tau_w(r)\big|_{r=a} = \mu \frac{\partial v_z}{\partial r}\bigg|_{r=a} = -\frac{\rho g a}{2\mu} + \frac{\rho g b^2}{2\mu a}$$