

## Chapter 7.1 Boundary Layer Theory

### 1. Historical Background and Boundary Layer Concepts

Before Prandtl 1905 Boundary layer theory:

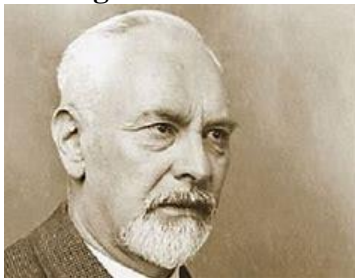
1. Mostly solutions of the linearized Navier Stokes equations for parallel and low Re flows as per Chapter 3 for which balance is for pressure and/or gravity and viscous forces, although few solutions include convection.
2. Advanced inviscid flow solutions for bodies of various shapes for which balance is for nonlinear inertia and pressure and/or gravity forces, i.e., potential, or inviscid rotational flows.
3. Empirical hydraulics

Potential flow theory was surprisingly accurate for streamlined bodies but suffered from **D'Alembert's paradox**.

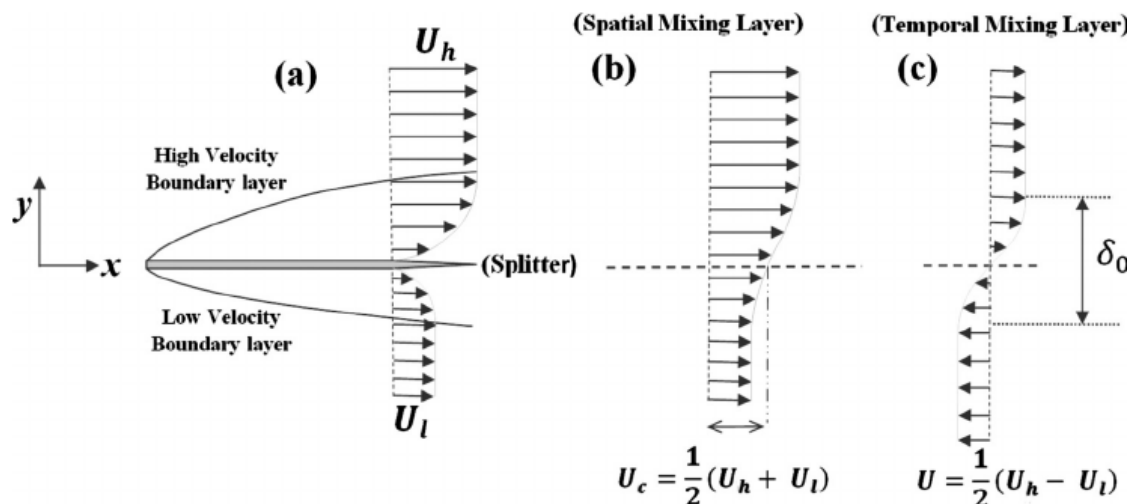
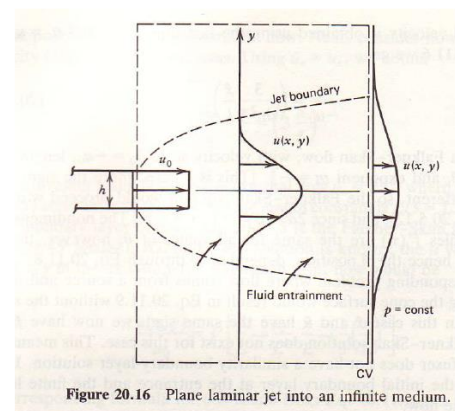
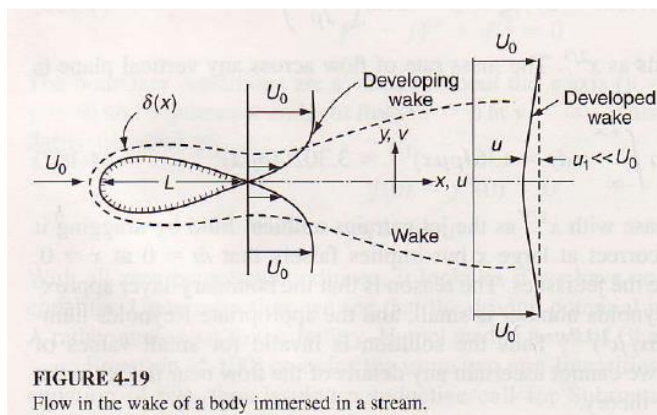
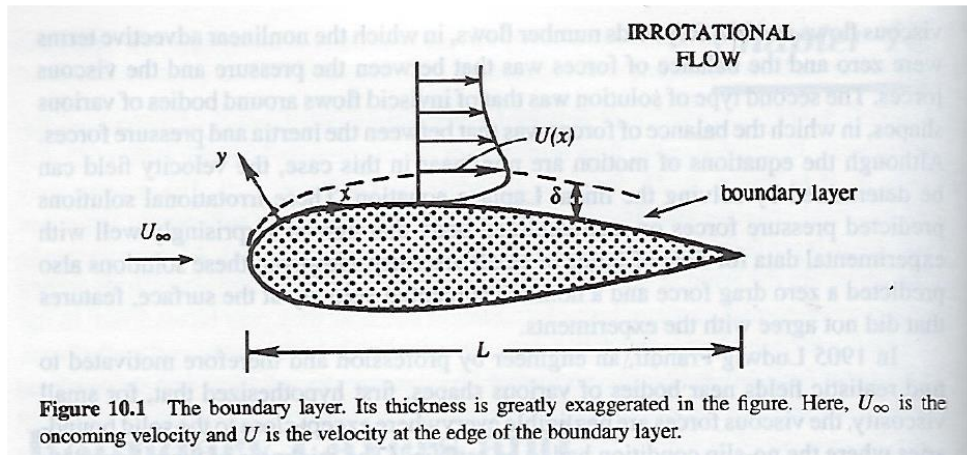


In fluid dynamics, d'Alembert's paradox (or the hydrodynamic paradox) is a contradiction reached in 1752 by French mathematician Jean le Rond d'Alembert. D'Alembert proved that – for incompressible and inviscid potential flow – the drag force is zero on a body moving with constant velocity relative to the fluid. Zero drag is in direct contradiction to the observation of substantial drag on bodies moving relative to fluids, such as air and water, especially at high velocities corresponding with high Reynolds numbers. It is a particular example of the reversibility paradox.

### Ludwig Prandtl



Ludwig Prandtl was a German fluid dynamics, physicist and aerospace scientist. He was a pioneer in the development of rigorous systematic mathematical analyses which he used for underlying the science of aerodynamics, which have come to form the basis of the applied science of aeronautical engineering. In the 1920s he developed the mathematical basis for the fundamental principles of subsonic aerodynamics in particular; and in general, up to and including transonic velocities. His studies identified the boundary layer, thin-airfoils, and lifting-line theories. The Prandtl number was named after him.



Consider high Re flow (1) around streamlined/slender body for which viscous effects are confined to a narrow boundary layer near the solid surface/wall or (2) for free shear flows, i.e. jets, wakes and mixing layers for which the vorticity is similarly confined to a narrow region. In both cases Prandtl's boundary layer theory is applicable. Formally uses concept of flow field regions and matched asymptotic expansions similarly as for Stokes flows.

## Boundary Layer Theory Approximations

### Momentum Scales

$$uu_x + vu_y = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu(u_{xx} + u_{yy})$$

$$uu_x \sim \frac{U^2}{L} = \nu u_{yy} \sim \nu \frac{U}{\delta^2}$$

$$\delta \sim \sqrt{\frac{\nu L}{U}} \text{ or } \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

$$Re \sim \sqrt{\frac{UL}{\nu}}$$

That is  $\delta$  is small for large  $Re$ . Note  $\sim$  indicates order of magnitude.

### Length Scales

Convection:  $L=Ut$  i.e.,  $t=L/U$

Viscous diffusion:  $\delta \sim \sqrt{\nu t}$  as per exact solutions NS

$$\delta \sim \sqrt{\frac{\nu L}{U}} \text{ or } \frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

### Time Scales

Convection:  $t=L/U$

Viscous diffusion:  $t = \sqrt{\frac{\nu L}{U^3}} \ll L/U$

$$\sqrt{\frac{\nu}{UL}} = \frac{1}{\sqrt{Re}} \ll 1$$

Note:  $\delta \sim \sqrt{\nu t} = \sqrt{\frac{\nu L}{U}}$ , i.e.  $\frac{\delta}{U} = \sqrt{\frac{\nu L}{U^3}} = t$

Boundary layer equations are derived from Navier Stokes equations under assumptions:

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

$$\frac{\partial^2}{\partial x^2} \ll \frac{\partial^2}{\partial y^2}$$

i.e. variation across the boundary layer are much larger than variation along the boundary layer.

Next consider continuity equation

$$u_x + v_y = 0$$

Since  $u \gg v$  and  $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$  and both terms are of equal order:  $\frac{\partial v}{\partial y} = \frac{v}{\delta} = \frac{\partial u}{\partial x} = \frac{U}{L}$ , i.e.,

$$v \sim \frac{\delta U}{L} \sim \frac{U}{\sqrt{Re}}$$

The order of magnitude of the pressure variations for which experimental data shows that the surface pressure is equivalent to the outer inviscid flow pressure which implies of order the inertia forces.

$$\frac{\partial p}{\partial x} \sim \rho u u_x$$

$$(p/L = \rho U^2 / L)$$

$$p - p_\infty \sim \rho U^2$$

The proper nondimensional variables in the boundary layer are, therefore:

$$x^* = x/L$$

$$y^* = \frac{y}{L} \sqrt{Re}$$

$$t^* = tU/L$$

$$u^* = u/U$$

$$v^* = \frac{v}{U} \sqrt{Re} = \frac{v}{\delta U/L}$$

$$p^* = \frac{p - p_0}{\rho U^2}$$

$$Re = UL/\nu$$

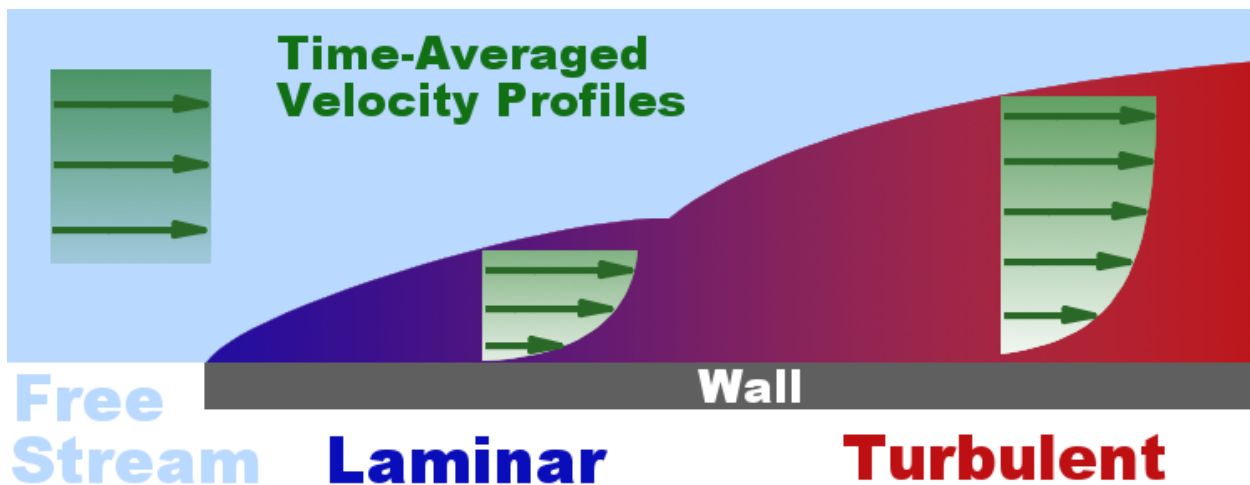
Note that  $\delta = \sqrt{\frac{\nu L}{U}}$  and that the distance and velocity across the boundary layer are stretched by  $\sqrt{Re}$ . Transformation of the Navier Stokes equations into nondimensional variables leads to Prandtl's boundary layer equations, which although retain streamwise convection neglect streamwise diffusion, and in addition show that the pressure variation across the boundary layer is also negligible such that it can be determined from the outer potential flow solution, and is imposed on the boundary layer.

Lastly the order of magnitude of the wall shear stress is:

$$\tau_w = \mu \frac{\partial u}{\partial y} = \mu U / \delta = (\mu U / L) \sqrt{Re} \quad (\text{where } \delta = L / \sqrt{Re})$$

$$C_f = \frac{\tau_w}{(\frac{1}{2})\rho U^2} = \frac{2\mu U}{\rho L U^2} \sqrt{Re} = \frac{2}{\sqrt{Re}}$$

Where different solutions have different numerical factors than 2.



In physics and fluid mechanics, a boundary layer is the layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant. In the Earth's atmosphere, the atmospheric boundary layer is the air layer near the ground affected by diurnal/daily heat, moisture, or momentum transfer to or from the surface. On an aircraft wing the boundary layer is the part of the flow close to the wing, where viscous forces distort the surrounding non-viscous flow.