3. A large tank of liquid under pressure is drained through a smoothly contoured nozzle of area A. The mass flow rate $\dot{m}$ is thought to depend on the nozzle area A , the liquid density $\rho$, the difference in height between the liquid surface and the nozzle $h$, the change in pressure $\Delta p$, and the gravitational acceleration g. Taking $\rho, A$, and $g$ as repeating variables, find an expression for the mass flow rate $\dot{m}$ as a function of the other parameters in the problem in terms of dimensionless Pi groups

| Quantity | Symbol | Dimensions |  |
| :---: | :---: | :---: | :---: |
|  |  | MLTE | FLT $\Theta$ |
| Length | $L$ | $L$ | $L$ |
| Area | A | $L^{2}$ | $L^{2}$ |
| Volume | Q | $L^{3}$ | $L^{3}$ |
| Velocity | V | $L T^{-1}$ | $L T^{-1}$ |
| Acceleration | $d V / d t$ | $L T^{-2}$ | $L T^{-2}$ |
| Speed of sound | $a$ | $L T^{-1}$ | $L T^{-1}$ |
| Volume flow | $Q$ | $L^{3} T^{-1}$ | $L^{3} T^{-1}$ |
| Mass flow | $\dot{m}$ | $M T^{-1}$ | $F T L^{-1}$ |
| Pressure, stress | $p, \sigma, \tau$ | $M L^{-1} T^{-2}$ | $F L^{-2}$ |
| Strain rate | P | $T^{-1}$ | $T^{-1}$ |
| Angle | $\theta$ | None | None |
| Angular velocity | $\omega, \Omega$ | $T^{-1}$ | $T^{-1}$ |
| Viscosity | $\mu$ | $M L^{-1} T^{-1}$ | $F T L^{-2}$ |
| Kinematic viscosity | $\nu$ | $L^{2} T^{-1}$ | $L^{2} T^{-1}$ |
| Surface tension | Y | $M T^{-2}$ | $F L^{-1}$ |
| Force | $F$ | MLT ${ }^{-2}$ | $F$ |
| Moment, torque | M | $M L^{2} T^{-2}$ | FL |
| Power | $P$ | $M L^{2} T^{-3}$ | $F L T^{-1}$ |
| Work, energy | $W, E$ | $M L^{2} T^{-2}$ | $F L$ |
| Density | $\rho$ | $M L^{-3}$ | $F T^{2} L^{-4}$ |
| Temperature | $T$ | $\Theta$ | $\Theta$ |
| Specific heat | $c_{p}, c_{v}$ | $L^{2} T^{-2} \Theta^{-1}$ | $L^{2} T^{-2} \Theta^{-1}$ |
| Specific weight | $\gamma$ | $M L^{-2} T^{-2}$ | $F L^{-3}$ |
| Thermal conductivity | , | $M L T^{-3} \boldsymbol{\Theta}^{-1}$ | $F T^{-1} \Theta^{-1}$ |
| Thermal expansion coefficient | $\beta$ | $\Theta^{-1}$ | $\Theta^{-1}$ |

## Solution 2:

Assumptions: the problem is only a function of the given dimensional variables.

$$
\begin{gather*}
\dot{m}=f(A, \rho, h, \Delta p, g) \\
n=6 \\
\dot{m}=\left\{M T^{-1}\right\} ; \quad A=\left\{L^{2}\right\} ; \rho=\left\{M L^{-3}\right\} ; h=\{L\} ; \Delta p=\left\{M L^{-1} T^{-2}\right\} ; \quad g=\left\{L T^{-2}\right\} \\
j=3 \rightarrow k=n-j=3 \tag{1}
\end{gather*}
$$

The repeating variables are $\rho, A$, and $g$; adding each remaining variable in turn, we find the Pi groups:

$$
\begin{gather*}
\Pi_{1}=\rho^{a} A^{b} g^{c} \dot{m}=\left\{\left(M L^{-3}\right)^{a}\left(L^{2}\right)^{b}\left(L T^{-2}\right)^{c}\left(M T^{-1}\right)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\} \\
a=-1 ; b=-5 / 4 ; c=-1 / 2 \\
\Pi_{1}=\frac{\dot{m}}{\rho A^{5 / 4} g^{1 / 2}}  \tag{2}\\
\Pi_{2}=\rho^{a} A^{b} g^{c} h=\left\{\left(M L^{-3}\right)^{a}\left(L^{2}\right)^{b}\left(L T^{-2}\right)^{c}(L)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\} \\
a=0 ; b=-1 / 2 ; c=0 \\
\Pi_{2}=\frac{h}{A^{1 / 2}}  \tag{2}\\
\Pi_{3}=\rho^{a} A^{b} g^{c} \Delta p=\left\{\left(M L^{-3}\right)^{a}\left(L^{2}\right)^{b}\left(L T^{-2}\right)^{c}\left(M L^{-1} T^{-2}\right)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\} \\
a=-1 ; b=-1 / 2 ; c=-1 \\
\Pi_{3}=\frac{\Delta p}{\rho A^{1 / 2} g} \tag{2}
\end{gather*}
$$

Thus the arrangement of the dimensionless variables is:

$$
\frac{\dot{m}}{\rho A^{5 / 4} g^{1 / 2}}=f\left(\frac{h}{A^{1 / 2}}, \frac{\Delta p}{\rho A^{1 / 2} g}\right)
$$

4. When small aerosol particles or microorganisms move through air or water, the Reynolds number is very small. The aerodynamic drag FD on an object in this condition is a function only of its speed V , some characteristic length scale L of the object, and fluid viscosity $\mu$. Use dimensional analysis to generate a relationship for FD as a function of the independent variables.


$$
\begin{gather*}
\stackrel{(1)}{F_{D}=M L T^{-2} ;} \mu=M L^{-1} T^{-1} ; L=\stackrel{(1)}{(1)} V=L T^{(1)} \Rightarrow n=4 \stackrel{(1)}{4} j=\stackrel{(0.5)}{3} \text { and } k=n-j=1
\end{gather*} \begin{aligned}
& \pi_{1}=F_{D} \mu^{a} L^{b} V^{c} \Rightarrow \pi_{1}=\left(M L T^{-2}\right)\left(M L^{-1} T^{-1}\right)^{a}(L)^{b}\left(L T^{-1}\right)^{c} \Rightarrow\left\{\begin{array}{l}
1+a=0 \\
1-a+b+c=0 \\
-2-2 a-c=0
\end{array}\right.  \tag{0.5}\\
& \left\{\begin{array}{l}
1+a=0 \\
1-a+b+c=0 \Rightarrow a=-1 ; b=-1 ; c=-1 \\
-2-a-c=0
\end{array}\right. \\
& \pi_{1}=\frac{F_{D}}{\mu L V}=\text { const. } \tag{1}
\end{aligned}
$$

1. Air discharge from a 2 in diameter nozzle and strikes a curved vane, which is in a vertical plane as shown in below figure. A stagnation tube connected to a water U-tube manometer is located in the free air jet. Determine the horizontal component of the force that the air jet exerts on the vane. Neglect the weight of the air and all friction.
$\gamma_{\text {water }}: 62.4 \mathrm{lb} / \mathrm{ft}^{3}$


## Solution 1:

KNOWN: Inlet and outlet diameter, Outlet flow angle, manometer height.
FIND:

- Force $R_{x}$ that requires to hold the plate stationary


## ASSUMPTIONS:

Neglect weight of air
Neglect friction loss

$$
\begin{equation*}
R_{x}=\dot{m} V_{1}-\dot{m} V_{2} \tag{1}
\end{equation*}
$$

Diameter of section (1) and (2) is same

$$
\begin{equation*}
\therefore\left|V_{1}\right|=\left|V_{2}\right| \tag{1}
\end{equation*}
$$

$V_{2}$ Direction is negative $x$ and has $\cos \left(30^{\circ}\right)$

$$
\begin{align*}
\therefore R_{x} & =\dot{m} V_{1}-\dot{m}\left(-V_{1} \cos 30^{\circ}\right) \\
& R_{x}=\dot{m} V_{1}\left(1+\cos 30^{\circ}\right) \tag{2}
\end{align*}
$$

Calculate $V_{1}$ using Bernoulli equation (Apply stagnation point and section(1))

$$
\begin{gathered}
\frac{P_{\text {stag }}}{\rho_{\text {air }}}+\frac{V_{\text {stag }}^{2}}{2}=\frac{P_{1}}{\rho_{\text {air }}}+\frac{V_{1}^{2}}{2} \\
\frac{P_{\text {stag }}}{\rho_{\text {air }}}=\frac{P_{1}}{\rho_{\text {air }}}+\frac{V_{1}^{2}}{2}
\end{gathered}
$$

From the manometer,

$$
\begin{gather*}
P_{\text {stag }}=P_{\text {atm }}+h \gamma_{\text {water }} \\
P_{1}=P_{\text {atm }} \\
\frac{P_{\text {atm }}+h \gamma_{\text {water }}}{\rho_{\text {air }}}=\frac{P_{\text {atm }}}{\rho_{\text {air }}}+\frac{V_{1}^{2}}{2} \\
\therefore \frac{V_{1}^{2}}{2}=\frac{h \gamma_{\text {water }}}{\rho_{\text {air }}}, \quad V_{1}=\sqrt{2 h \frac{\gamma_{\text {water }}}{\rho_{\text {air }}}} \tag{2}
\end{gather*}
$$

Plug in $V_{1}$ to the $R_{x}$

$$
\begin{gather*}
R_{x}=\dot{m} V_{1}\left(1+\cos 30^{\circ}\right)=\rho_{\text {air }} \frac{\pi}{4}\left(\frac{2}{12}\right)^{2} V_{1}^{2}\left(1+\cos 30^{\circ}\right) \\
R_{x}=\rho_{\text {air }} \frac{\pi}{4}\left(\frac{2}{12}\right)^{2} 2 h \frac{\gamma_{\text {water }}}{\rho_{\text {air }}}\left(1+\cos 30^{\circ}\right) \\
R_{x}=\frac{\pi h}{2}\left(\frac{2}{12}\right)^{2} \gamma_{\text {water }}\left(1+\cos 30^{\circ}\right) \\
R_{x}=\frac{\pi(7 / 12)}{2}\left(\frac{2}{12}\right)^{2} 62.4\left(1+\cos 30^{\circ}\right) \\
R_{x}=2.963 l b \tag{2}
\end{gather*}
$$

3. The viscous, incompressible flow between the parallel plates shown in Figure is caused by both the motion of the bottom plate and a constant pressure gradient $\frac{\partial p}{\partial x}$. Assuming steady, 2D, and parallel flow and using differential analysis: (a) Show that the flow is fully developed using continuity equation; (b) Find the velocity profile $u(y)$ using Navier-Stokes equations with appropriate boundary conditions; (c) Find wall shear stress at bottom wall; and (d) Find the flow rate (hint: $Q=\int \underline{V} \cdot \underline{n} d A$ and assume constant width w). Explicitly state all assumptions.

$\rho g_{x}-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}+\frac{\partial^{2} u}{\partial z^{2}}\right)=\rho\left(u \frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}+w \frac{\partial u}{\partial z}\right)$
$\rho g_{y}-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^{2} v}{\partial x^{2}}+\frac{\partial^{2} v}{\partial y^{2}}+\frac{\partial^{2} v}{\partial z^{2}}\right)=\rho\left(v \frac{\partial v}{\partial t}+u \frac{\partial v}{\partial x}+v \frac{\partial v}{\partial y}+w \frac{\partial v}{\partial z}\right)$
$\rho g_{z}-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^{2} w}{\partial x^{2}}+\frac{\partial^{2} w}{\partial y^{2}}+\frac{\partial^{2} w}{\partial z^{2}}\right)=\rho\left(w \frac{\partial w}{\partial t}+u \frac{\partial w}{\partial x}+v \frac{\partial w}{\partial y}+w \frac{\partial w}{\partial z}\right)$

## Solution 3:

KNOWN: Flow condition, Boundary condition
FIND:

- Show that the flow is fully developed
- velocity profile $u(y)$
- Find wall shear stress at bottom wall
- Find the flow rate


## ASSUMPTIONS:

- steady: $\frac{\partial}{\partial t}=0$
-2D flow: $w=0 ; \quad \frac{\partial}{\partial z}=0$
- Parallel flow: $v=w=0$
$-\frac{\partial p}{\partial x}=$ constant

ANALYSIS:
(a) Continuity

$$
\begin{gather*}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \\
\frac{\partial u}{\partial x}+0+0=0 \rightarrow \frac{\partial u}{\partial x}=0 \rightarrow \quad \text { fully developed } \tag{1}
\end{gather*}
$$

(b) $x$-momentum equation

$$
\begin{gather*}
\rho\left(\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+v \frac{\partial u}{\partial y}+w \frac{\partial u}{\partial z}\right)=\rho g_{x}-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}+\frac{\partial^{2} u}{\partial z^{2}}\right) \\
0=-\frac{\partial P}{\partial x}+\mu \frac{\partial^{2} u}{\partial y^{2}} \\
\frac{\partial^{2} u}{\partial y^{2}}=\frac{1}{\mu} \frac{\partial p}{\partial x} \tag{1.5}
\end{gather*}
$$

Integrate twice

$$
\begin{equation*}
u(y)=\frac{1}{2 \mu} \frac{\partial p}{\partial x} y^{2}+c_{1} y+c_{2} \tag{1.5}
\end{equation*}
$$

Apply boundary conditions

$$
\begin{gathered}
y=0 \rightarrow u=U \quad \rightarrow \quad c_{2}=U \\
y=b \rightarrow u=0 \rightarrow 0=\frac{1}{2 \mu}\left(\frac{\partial p}{\partial x}\right) b^{2}+c_{1} b+U \rightarrow \quad c_{1}=-\frac{1}{2 \mu}\left(\frac{\partial p}{\partial x}\right) b-\frac{U}{b}
\end{gathered}
$$

Therefore,

$$
u(y)=\frac{1}{2 \mu}\left(\frac{\partial p}{\partial x}\right)\left(y^{2}-b y\right)+U\left(1-\frac{y}{b}\right)
$$

(c) Shear stress at bottom wall

$$
\begin{gather*}
\tau=\mu \frac{d u}{d y}=\frac{1}{2}\left(\frac{\partial p}{\partial x}\right)(2 y-b)-\frac{U}{b} \mu \\
\tau_{\text {wall }}=\tau(0)=-\frac{b}{2}\left(\frac{\partial p}{\partial x}\right)-\frac{U}{b} \mu \tag{1.5}
\end{gather*}
$$

(d) Flow rate

$$
\begin{gather*}
Q=\int \underline{V} \cdot \underline{n} d A=w \int_{0}^{b} u(y) d y=\frac{w}{2 \mu}\left(\frac{\partial p}{\partial x}\right)\left(\frac{1}{3} y^{3}-\frac{b}{2} y^{2}\right)_{0}^{b}+w U\left(y-\frac{1}{2 b} y^{2}\right)_{0}^{b} \\
Q=\frac{w}{2 \mu}\left(\frac{\partial p}{\partial x}\right)\left(\frac{1}{3} b^{3}-\frac{1}{2} b^{3}\right)+w U\left(b-\frac{1}{2} b\right)=-\frac{w}{12 \mu}\left(\frac{\partial p}{\partial x}\right) b^{3}+\frac{w}{2} U b \tag{1.5}
\end{gather*}
$$

Consider natural convection in a rotating, fluid-filled enclosure. The average wall shear stress $\tau$ in the enclosure is assumed to be a function of rotation rate $\Omega$, enclosure height $H$, density $\rho$, temperature difference $\Delta T$, viscosity $\mu$, thermal expansion coefficient $\beta$, and gravity acceleration $g$. Rewrite this relationship as a dimensionless function. Use the following repeating variables: $\rho, H, \Omega$, and $\beta$.

| Quantity | Symbol | Dimensions |  |
| :---: | :---: | :---: | :---: |
|  |  | MLTE | FLTE |
| Length | $L$ | $L$ | $L$ |
| Area | A | $L^{2}$ | $L^{2}$ |
| Volume | Q | $L^{3}$ | $L^{3}$ |
| Velocity | V | $L T^{-1}$ | $L T^{-1}$ |
| Acceleration | $d V / d t$ | $L T^{-2}$ | $L T^{-2}$ |
| Speed of sound | $a$ | $L T^{-1}$ | $L T^{-1}$ |
| Volume flow | $Q$ | $L^{3} T^{-1}$ | $L^{3} T^{-1}$ |
| Mass flow | $\dot{m}$ | $M T^{-1}$ | FTL ${ }^{-1}$ |
| Pressure, stress | $p, \sigma, \tau$ | $M L^{-1} T^{-2}$ | $F L^{-2}$ |
| Strain rate | $\dot{\epsilon}$ | $T^{-1}$ | $T^{-1}$ |
| Angle | $\theta$ | None | None |
| Angular velocity | $\omega, \Omega$ | $T^{-1}$ | $T^{-1}$ |
| Viscosity | $\mu$ | $M L^{-1} T^{-1}$ | $F T L^{-2}$ |
| Kinematic viscosity | $\nu$ | $L^{2} T^{-1}$ | $L^{2} T^{-1}$ |
| Surface tension | Y | $M T^{-2}$ | $F L^{-1}$ |
| Force | F | MLT ${ }^{-2}$ | F |
| Moment, torque | M | $M L^{2} T^{-2}$ | $F L$ |
| Power | $P$ | $M L^{2} T^{-3}$ | $F L T^{-1}$ |
| Work, energy | $W, E$ | $M L^{2} T^{-2}$ | $F L$ |
| Density | $\rho$ | $M L^{-3}$ | $F T^{2} L^{-4}$ |
| Temperature | $T$ | $\Theta$ | $\Theta$ |
| Specific heat | $c_{p}, c_{v}$ | $L^{2} T^{-2} \Theta^{-1}$ | $L^{2} T^{-2} \Theta^{-1}$ |
| Specific weight | $\gamma$ | $M L^{-2} T^{-2}$ | $F L^{-3}$ |
| Thermal conductivity | $k$ | $M L T^{-3} \Theta^{-1}$ | $F T^{-1} \Theta^{-1}$ |
| Thermal expansion coefficient | $\beta$ | $\Theta^{-1}$ | $\Theta^{-1}$ |

KNOWN: dimensional parameters
FIND: Pi groups

ASSUMPTIONS: the problem is only a function of the given dimensional variables ANALYSIS:

$$
\begin{gathered}
\tau=f(\rho, H, \Omega, \beta, \mu, \Delta T, g) \\
n=8 \\
\tau=\left\{M L^{-1} T^{-2}\right\} ; \rho=\left\{M L^{-3}\right\} ; \quad H=\{L\} ; \quad \Omega=\left\{T^{-1}\right\} \\
\beta=\left\{\Theta^{-1}\right\} ; \mu=\left\{M L^{-1} T^{-1}\right\} ; \Delta T=\{\Theta\} ; g=\left\{L T^{-2}\right\} \\
j=4 \rightarrow k=n-j=4
\end{gathered}
$$

The repeating variables are $\rho, H, \Omega$, and $\beta$; adding each remaining variable in turn, we find the Pi groups:

$$
\begin{gathered}
\Pi_{1}=\rho^{a} H^{b} \Omega^{c} \beta^{d} r=\left\{\left(M L^{-3}\right)^{a}(L)^{b}\left(T^{-1}\right)^{c}\left(\Theta^{-1}\right)^{d}\left(M L^{-1} T^{-2}\right)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\}+0.5 \\
a=-1 ; b=-2 ; c=-2 ; d=0 \\
\Pi_{1}=\frac{\tau}{\rho H^{2} \Omega^{2}}+0.5 \\
\begin{array}{r}
\Pi_{2}=\rho^{a} H^{b} \Omega^{c} \beta^{d} \mu=\left\{\left(M L^{-3}\right)^{a}(L)^{b}\left(T^{-1}\right)^{c}\left(\Theta^{-1}\right)^{d}\left(M L^{-1} T^{-1}\right)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\}+0.5 \\
a=-1 ; b=-2 ; c=-1 ; d=0
\end{array} \\
\Pi_{2}=\frac{\mu}{\rho H^{2} \Omega}+0.5 \\
\Pi_{3}=\rho^{a} H^{b} \Omega^{c} \beta^{d} \Delta T=\left\{\left(M L^{-3}\right)^{a}(L)^{b}\left(T^{-1}\right)^{c}\left(\Theta^{-1}\right)^{d}(\Theta)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\} \\
a=0 ; b=0 ; c=0 ; d=1 \\
\Pi_{3}=\beta \Delta T \quad+0.5 \\
\Pi_{4}=\rho^{a} H^{b} \Omega^{c} \beta^{d} g=\left\{\left(M L^{-3}\right)^{a}(L)^{b}\left(T^{-1}\right)^{c}\left(\Theta^{-1}\right)^{d}\left(L T^{-2}\right)\right\}=\left\{M^{0} L^{0} T^{0} \Theta^{0}\right\}+0.5
\end{gathered}
$$

