# ME:5160 (58:160) Intermediate Mechanics of Fluids 

Fall 2023 - HW7 Solution
*P4.91 Analyze fully developed laminar pipe flow for a power-law fluid, $\tau=\mathrm{C}\left(d v_{z} / d r\right)^{\mathrm{n}}$, for $n$ $\neq 1$, as in Prob. P1.46. (a) Derive an expression for $v_{z}(r)$. (b) For extra credit, plot the velocity profile shapes for $n=0.5,1$, and 2. [Hint: In Eq. (4.140), replace $\mu\left(\mathrm{d} v_{z} / \mathrm{d} r\right)$ by $\tau$.]
Solution: (a) For the power-law fluid, Eq. (4.140) becomes

$$
\frac{1}{r} \frac{d}{d r}(r \tau)=\frac{1}{r} \frac{d}{d r}\left[r C\left(\frac{d v_{z}}{d r}\right)^{n}\right]=\frac{d p}{d z}
$$

Multiply by $r$ and integrate once:

$$
r C\left(\frac{d v_{z}}{d r}\right)^{n}=\frac{d p}{d z} \frac{r^{2}}{2}+B
$$

where $B$ is a constant of integration. Divide by $r$, take the $n^{\text {th }}$ root, and integrate again:

$$
\frac{d v_{z}}{d r}=\left[\left(\frac{1}{2 C} \frac{d p}{d z}\right) r+\frac{B}{r}\right]^{1 / n}=\left[\left(\frac{1}{2 C} \frac{d p}{d z}\right) r\right]^{1 / n}
$$

since $B$ must be zero, to avoid a logarithmic singularity at the origin. Integrate once more:

$$
\begin{aligned}
& v_{z}=\left(\frac{n}{n+1}\right)\left(\frac{1}{2 C} \frac{d p}{d z}\right)^{1 / n} r^{(n+1) / n}+A \\
& \text { At } r=R, \quad v_{z}=0, \text { hence } A=-\left(\frac{n}{n+1}\right)\left(\frac{1}{2 C} \frac{d p}{d z}\right)^{1 / n} R^{(n+1) / n}
\end{aligned}
$$

The final solution, analogous to Eq. (4.137) for newtonian Poiseuille flow, is

$$
v_{z}=\left(\frac{n}{n+1}\right)\left(\frac{-1}{2 C} \frac{d p}{d z}\right)^{1 / n}\left[R^{(n+1) / n}-r^{(n+1) / n}\right] \quad \text { Ans. }(a)
$$

For $n=1$, for which $C=\mu$, this reduced to Eq. (4.141) for Poiseuille flow.
(b) To plot as a comparison, let $[(-d p / d z) / 2 C]$ equal unity and plot $v_{z}$ for $n=0.5,1$, and 2 :


Note that $n=0.5$, which resembles a pseudoplastic fluid, Fig. 1.9 , is very flat, while $n=2$, similar to a dilatant fluid, Fig. 1.9 , is very steep.

P4.94 A long solid cylinder rotates steadily in a very viscous fluid, as in Fig. P4.94.

Assuming laminar flow, solve the Navier-Stokes equation in polar coordinates to determine the resulting velocity distribution. The fluid is at rest far from the cylinder. [HINT: the cylinder does not induce any radial motion.]


Fig. P4.94

Solution: We already have the useful hint that $v_{\mathrm{r}}=0$. Continuity then tells us that $(1 / r) \partial v_{\theta} / \partial \theta=0$, hence $v_{\theta}$ does not vary with $\theta$. Navier-Stokes then yields the flow. From Eq. D.6, the tangential momentum relation, with $\partial p / \partial \theta=0$ and $v_{\theta}=f(r)$, we obtain Eq. (4.143):

$$
\begin{aligned}
& \frac{1}{r} \frac{d}{d r}\left(r \frac{d v_{\theta}}{d r}\right)=\frac{v_{\theta}}{r^{2}}, \quad \text { Solution }: v_{\theta}=C_{1} r+\frac{C_{2}}{r} \\
& \text { As } r \rightarrow \infty, v_{\theta} \rightarrow 0, \text { hence } C_{1}=0 \\
& \text { At } r=R, \quad v_{\theta}=\Omega R=\frac{C_{2}}{R} ; C_{2}=\Omega R^{2} ; \text { Finally, } v_{\theta}=\frac{\Omega R^{2}}{r} \quad \text { Ans. }
\end{aligned}
$$

Rotating a cylinder in a large expanse of fluid sets up (eventually) a potential vortex flow.

P5.28 A simply supported beam of diameter $D$, length $L$, and modulus of elasticity $E$ is subjected to a fluid crossflow of velocity $V$, density $\rho$, and viscosity $\mu$. Its center deflection $\delta$ is assumed to be a function of all these variables. (a) Rewrite this proposed function in dimensionless form. (b) Suppose it is known that $\delta$ is independent of $\mu$, inversely proportional to $E$, and dependent only upon $\rho V$ ${ }^{2}$, not $\rho$ and $V$ separately. Simplify the dimensionless function accordingly

Solution: Establish the variables and their dimensions:

$$
\begin{array}{cccccccc}
\delta=\mathrm{fcn}\left(\begin{array}{ccc}
\rho
\end{array},\right. & \mathrm{D}, & \mathrm{~L}, & \mathrm{E} \\
\{\mathrm{~L}\} & \left\{\mathrm{M} / \mathrm{L}^{3}\right\} & \{\mathrm{L}\} & \{\mathrm{L}\} & \left\{\mathrm{M} / \mathrm{LT}^{2}\right\} & \{\mathrm{L} / \mathrm{T}\} & (\mathrm{M} / \mathrm{LT}\}
\end{array}
$$

Then $n=7$ and $j=3$, hence we expect $\mathrm{n}-\mathrm{j}=7-3=4$ Pi groups, capable of various arrangements and selected by the writer, as follows (a):

$$
\text { Well-posed final result: } \frac{\delta}{\mathbf{L}}=\mathbf{f c n}\left(\frac{\mathbf{L}}{\mathbf{D}}, \frac{\rho \mathbf{V} \mathbf{D}}{\mu}, \frac{\mathbf{E}}{\rho \mathbf{V}^{2}}\right) \text { Ans. (a) }
$$

(b) If $\mu$ is unimportant and $\delta$ proportional to $E^{-1}$, then the Reynolds number ( $\rho \mathrm{VD} / \mu$ ) drops out, and we have already cleverly combined E with $\rho \mathrm{V}^{2}$, which we can now slip out and turn upside down:

If $\mu$ drops out and $\delta \propto \frac{1}{\mathrm{E}}, \quad$ then $\frac{\delta}{\mathrm{L}}=\frac{\rho \mathrm{V}^{2}}{\mathrm{E}} \mathrm{fcn}\left(\frac{\mathrm{L}}{\mathrm{D}}\right)$,

$$
\text { or: } \frac{\delta \mathbf{E}}{\rho \mathbf{V}^{2} \mathbf{L}}=\mathbf{f c n}\left(\frac{\mathbf{L}}{\mathbf{D}}\right) \text { Ans. (b) }
$$

P5.20
A fixed cylinder of diameter $D$ and length $L$, immersed in a stream flowing normal to its axis at velocity $U$, will experience zero average lift. However, if the cylinder is rotating at angular velocity $\Omega$, a lift force $F$ will arise. The fluid density $\rho$ is important, but viscosity is secondary and can be neglected. Formulate this lift behavior as a dimensionless function.

Solution: No suggestion was given for the repeating variables, but for this type of problem (force coefficient, lift coefficient), we normally choose ( $\rho, U, D$ ) for the task. List the dimensions:
D
\{L\}
$L$
U
$\Omega$
F
$\left\{\mathrm{T}^{-1}\right\}$
$\left\{\mathrm{MLT}^{-2}\right.$ \}
$\left\{\mathrm{ML}^{-3}\right\}$

There are three dimensions (MLT), which we knew when we chose ( $\rho, U, D$ ). Combining these three, separately, with $F, \Omega$, and $L$, we find this dimensionless function:

$$
\frac{F}{\rho U^{2} D^{2}}=\operatorname{fcn}\left(\frac{\Omega D}{U}, \frac{L}{D}\right)
$$

Ans.

This is a correct solution for Chapter 5, but in Chapter 8 we will use the "official" function, with extra factors of (1/2):

$$
\frac{F}{(1 / 2) \rho U^{2} L D}=\operatorname{fcn}\left(\frac{\Omega D}{2 U}, \frac{L}{D}\right)
$$

P5.68 For the rotating-cylinder function of Prob. P5.20, if $L \gg D$, the problem can be reduced to only two groups, $F /\left(r U^{2} L D\right)$ versus $(\mathrm{W} D / U)$. Here are experimental data for a cylinder 30 cm in diameter and 2 m long, rotating in sea-level air, with $U=25 \mathrm{~m} / \mathrm{s}$.

| W, rev/min | 0 | 3000 | 6000 | 9000 | 12000 | 15000 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $F$, N | 0 | 850 | 2260 | 2900 | 3120 | 3300 |

(a) Reduce this data to the two dimensionless groups and make a plot. (b) Use this plot to predict the lift of a cylinder with $D=5 \mathrm{~cm}, L=80 \mathrm{~cm}$, rotating at $3800 \mathrm{rev} / \mathrm{min}$ in water at $U=4$ $\mathrm{m} / \mathrm{s}$.

Solution: (a) In converting the data, the writer suggests using W in $\mathrm{rad} / \mathrm{s}$, not rev $/ \mathrm{min}$. For sealevel air, $r=1.2255 \mathrm{~kg} / \mathrm{m}^{3}$. Take, for example, the first data point, $\mathrm{W}=3000 \mathrm{rpm} \mathrm{x}(2 \mathrm{p} / 60)=$ $314 \mathrm{rad} / \mathrm{s}$, and $F=850 \mathrm{~N}$.

$$
\Pi_{1}=\frac{F}{\rho U^{2} L D}=\frac{850}{(1.2255)(25)^{2}(2.0 m)(0.3 m)}=1.85 ; \Pi_{2}=\frac{\Omega D}{U}=\frac{(314)(0.3)}{25}=3.77
$$

Do this for the other four data points, and plot as follows. Ans.(a)

(b) For water, take $r=998 \mathrm{~kg} / \mathrm{m}^{3}$. The new data are $D=5 \mathrm{~cm}, L=80 \mathrm{~cm}, 3800 \mathrm{rev} / \mathrm{min}$ in water at $U=4 \mathrm{~m} / \mathrm{s}$. Convert $3800 \mathrm{rev} / \mathrm{min}=398 \mathrm{rad} / \mathrm{s}$. Compute the rotation Pi group:

$$
\Pi_{2}=\frac{\Omega D}{U}=\frac{(398 \mathrm{rad} / \mathrm{s})(0.05 \mathrm{~m})}{4 \mathrm{~m} / \mathrm{s}}=4.97
$$

Read the chart for $\mathrm{P}_{1}$. The writer reads $\mathrm{P}_{1} \approx 2.8$. Thus we estimate the water lift force:

$$
F=\Pi_{1} \rho U^{2} L D=(2.8)(998)(4)^{2}(0.8 m)(0.05 m) \approx 1788 N \approx \mathbf{1 8 0 0} N \text { Ans. }(b)
$$

C5.3 Reconsider the fully-developed drain-ing vertical oil-film problem (see Fig. P4.80) as an exercise in dimensional analysis. Let the vertical velocity be a function only of distance from the plate, fluid properties, gravity, and film thickness. That is, $w=\mathrm{fcn}(x, \rho, \mu, g, \delta)$.
(a) Use the Pi theorem to rewrite this function in terms of dimensionless parameters.
(b) Verify that the exact solution from Prob. 4.80 is consistent with your result in part (a).


Solution: There are $n=6$ variables and $j=3$ dimensions (M, L, T), hence we expect only $n-j$ $=6-3=3$ Pi groups. The author selects $(\rho, g, \delta)$ as repeating variables, whence

$$
\Pi_{1}=\frac{w}{\sqrt{g \delta}} ; \quad \Pi_{2}=\frac{\mu}{\rho \sqrt{g \delta^{3}}} ; \quad \Pi_{3}=\frac{x}{\delta}
$$

Thus the expected function is

$$
\frac{w}{\sqrt{g \delta}}=f c n\left(\frac{\mu}{\rho \sqrt{g \delta^{3}}}, \frac{x}{\delta}\right) \quad \text { Ans. (a) }
$$

(b) The exact solution from Problem 4.80 can be written in just this form:

$$
w=\frac{\rho g x}{2 \mu}(x-2 \delta), \quad \text { or: } \quad \frac{w}{\sqrt{g \delta}} \frac{\mu}{\rho \sqrt{g \delta^{3}}}=\frac{1}{2} \frac{x}{\delta}\left(\frac{x}{\delta}-2\right)
$$

Yes, the two forms of dimensionless function are the same. Ans. (b)

