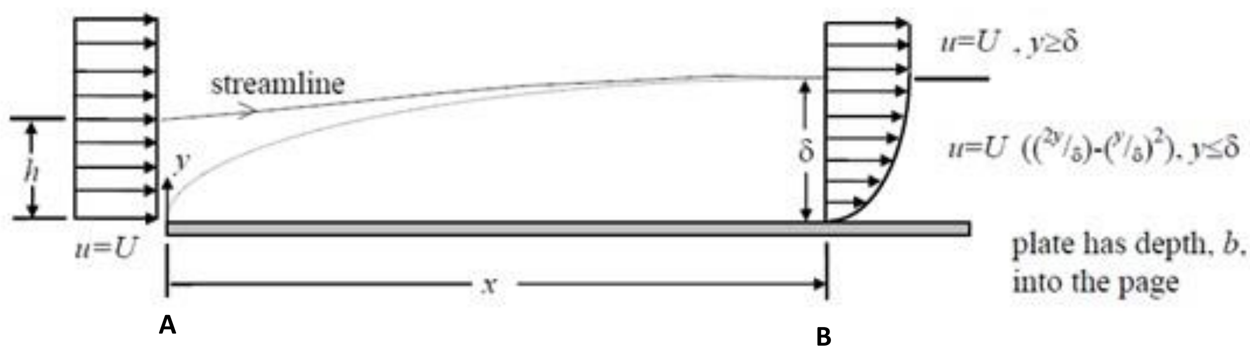
Name: ------------------ **Final Exam** **Time: 120 minutes**

ME:5160 **Fall 2023**

The exam is closed book and closed notes.

1. An incompressible, viscous fluid with density, ρ, flows past a solid flat plate which has a depth, b, into the page. The flow initially has a uniform velocity U before contacting the plate. The velocity profile at location x is estimated to have a parabolic shape, $u = U\left[\left(\frac{2y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$, for $y \leq \delta$ and $u = U$ for $y \geq \delta$ where δ is the boundary layer thickness. (a) Write the continuity equation and determine the upstream height from the plate, h, of a streamline which has a height, δ, at the downstream location. Express your answer in terms of δ. (b) Determine the force the fluid exerts on the plate over the distance x. Express your answer in terms of $\rho, U, b,$ and δ. You may assume that the pressure everywhere is atmospheric pressure.



Continuity equation: $-\frac{d}{dt}\int_{CV}\rho d\forall = \int_{CS}\rho \underline{V_R}\cdot \underline{n}dA$

Momentum equation: $\sum F = \frac{d}{dt}\int_{CV}\rho \underline{V} d\forall + \int_{CS}\rho \underline{V}\underline{V_R}\cdot \underline{n}dA$

Integration formula: $\int y^n\, dy = \frac{y^{n+1}}{n+1} + C$

Hint: write continuity and momentum equation between sections A (constant velocity) and B.

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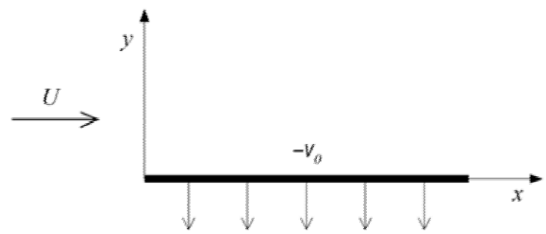
2. Consider steady flow of a Newtonian fluid of density ρ and viscosity μ at velocity U (at $y = \infty$) past an infinite plane ($y = 0$) as shown in the Figure below. The flow is 2D, incompressible, laminar, and gravity effects are negligible. The plate is porous and has constant suction $v = -v_0$. Assume that pressure is constant and $v = -v_0$ everywhere in the flow. **Simplify the continuity and Navier Stokes equations** and apply boundary conditions to find the velocity distribution u (y). (**Hint:** for solving differential equation $a\frac{\partial^2 u}{\partial y^2} + b\frac{\partial u}{\partial y} + c = 0$ with constant coefficients a, b, and c, assume that the solution is $e^{\lambda y}$ and find λ.)



Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Incompressible Navier-Stokes Equations in Cartesian Coordinates:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

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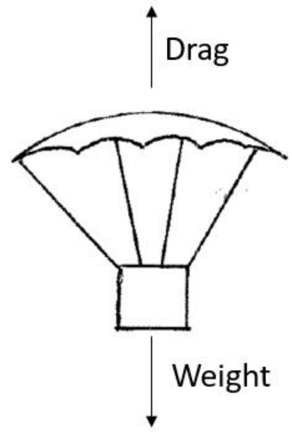
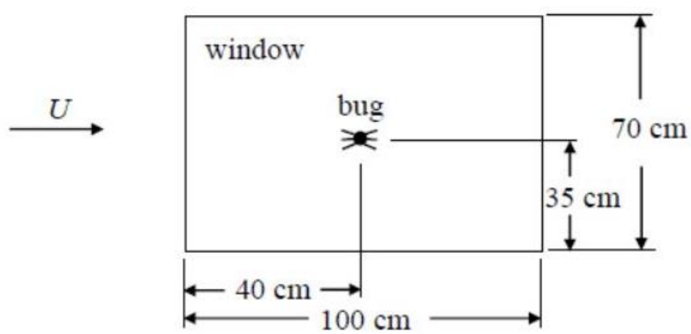
3. A parachute of a new design is tested in standard air (ρ=1.2 kg/m^3 and μ=1.8E-5 kg/m-s) with a total weight of the load and parachute of 200 N. The diameter for the tested prototype is 5 m. The results showed that the parachute reaches a constant velocity of 3 m/s. (a) Use the prototype data and find the drag coefficient for the parachute. (b) If you are to repeat the experiment for a 2.5 times smaller model using Reynolds similarity, and the weight of the model parachute comes out as 40 N, how much load do you need to add?

(**Hint:** $C_D = \frac{Drag}{\frac{1}{2}\rho V^2 A}$, $Re = \frac{\rho V D}{\mu}$)



4. A small bug rests on the outside of a car side window as shown in the Figure below. The surrounding air has a density of ρ=1.2 kg/m^3 and viscosity of μ=1.8E-5 kg/m-s. Assume that the flow can be approximated as flat plate flow with no pressure gradient and the start of the boundary layer begins at the leading edge of the window. (a) Assuming that the flow is turbulent where the bug is, determine the minimum speed at which the bug will be sheared off of the car window if the bug can resist a shear stress of up to 1 N/m^2. (b) Confirm the turbulent flow assumption. (c) What is the total skin friction drag acting on the window at this speed?

(Turbulent BL: $c_f = \frac{2\tau_w}{\rho U^2} \sim \frac{0.027}{Re_x^{1/7}}, C_D \sim \frac{0.031}{Re_L^{1/7}}$)



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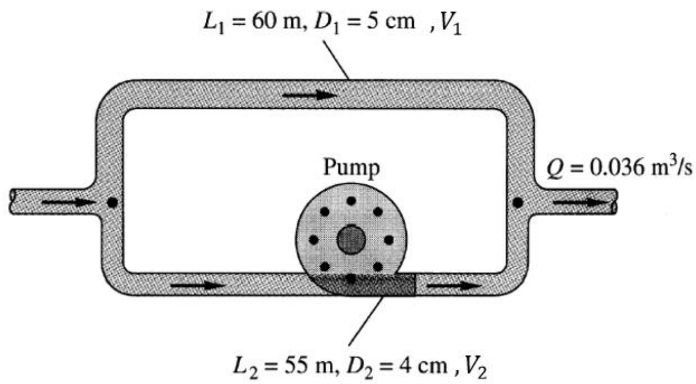
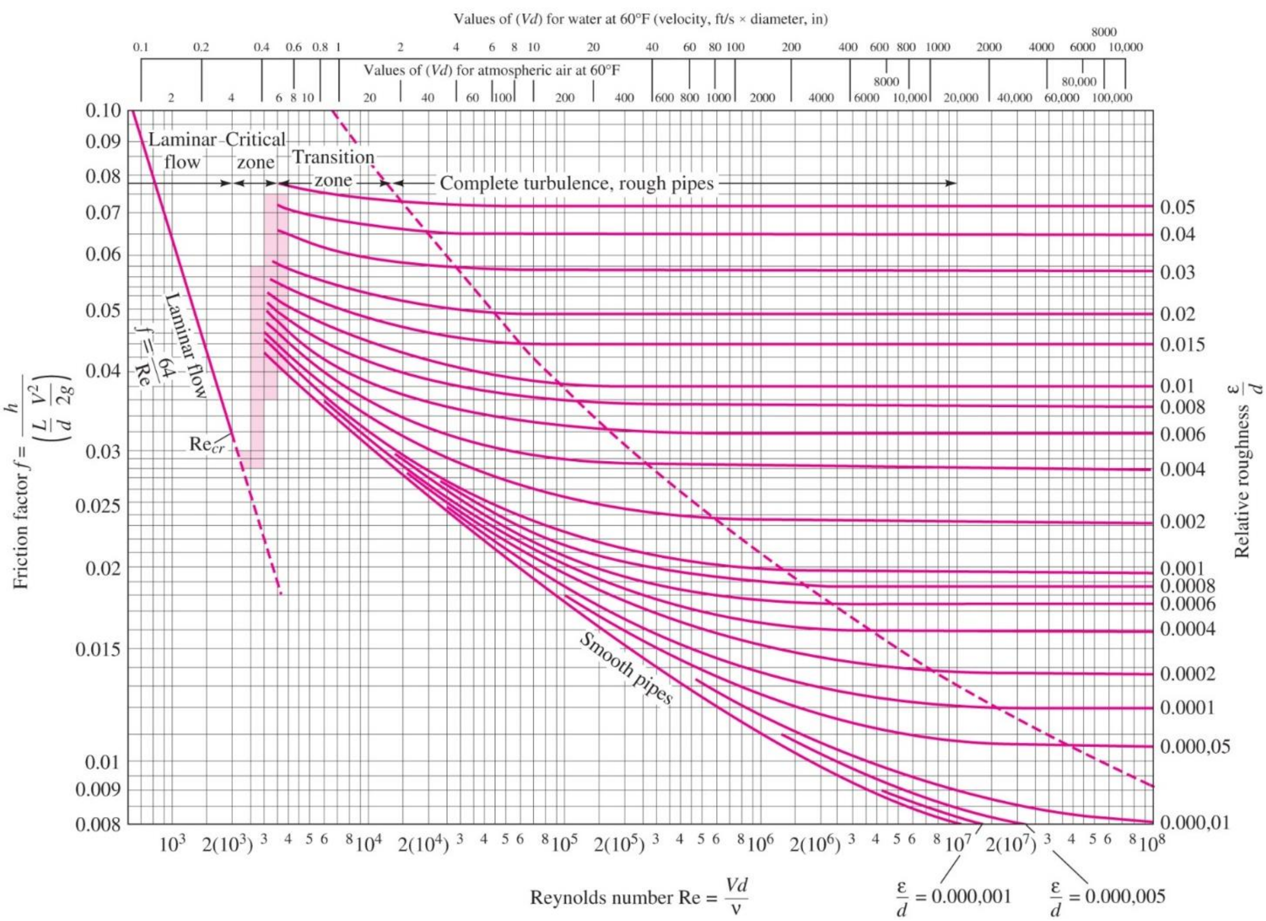
5. The parallel galvanized-iron pipe system ($\varepsilon = 0.15\ mm$) delivers water at 20°C ($\rho = 998\ kg/m^3$ and $\mu = 0.001\ kg/m \cdot s$) with a total flow rate of 0.036 m^3/s. (a) Find out the relation between V_1 and V_2. If the pump is wide open and not running, with a loss coefficient of K=1.5, (b) determine the velocity in each pipe (V_1 and V_2). Use $f_1 = f_2 = 0.02$ for your initial guess and do one iteration on the Moody's diagram.

(Hint : $h_f = f \frac{L}{d}\frac{V^2}{2g}$, for pipes in parallel: $Q = Q_1 + Q_2$, $\Delta h_{f_1} = \Delta h_{f_2}$, for pipes in series: $Q = Q_1 = Q_2$, $\Delta h = \Delta h_1 + \Delta h_2$)





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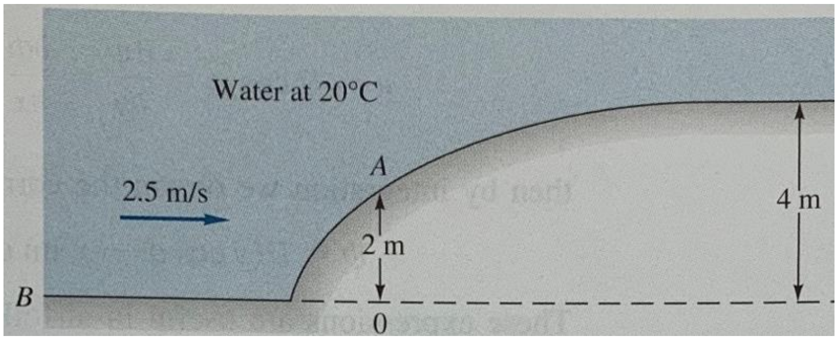
6. The bottom of a river has a 4 m high bump that approximates a Rankine half-body, as in Figure. The pressure at point B on bottom is 130 kPa, and the river velocity is 2.5 m/s. Use inviscid theory to estimate the water (a) velocity and (b) pressure at point A on the bump, which is 2 m above point B. ($\sin^2\theta + \cos^2\theta = 1; \rho = 998\ kg/m^3; \gamma = 9790\ N/m^3$)

Rankine half body equations:



$\Psi = Ur\sin\theta + m\theta;\ m = Ua$

$v_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta};\ v_\theta = -\frac{\partial\Psi}{\partial r}$

$r = \frac{m(\pi-\theta)}{U\sin\theta}$ (on surface)

Bump downstream height = πa

Hint:

(a) at point A, $\theta = \pi/2$ and $a = 4/\pi$

(b) Bernoulli's equation:

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

Solution 1

(a) Continuity Equation:

$$\int_0^h U\, b\, dy = \int_0^\delta u(y)\, b\, dy \qquad \boxed{+2}$$

$$bhU = \int_0^\delta bU\left[\left(\frac{2y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy$$

$$= bU\left(\frac{y^2}{\delta} - \frac{y^3}{3\delta^2}\right)_0^\delta = bU\left(\frac{2\delta}{3}\right)$$

$$bhU = \frac{2}{3} bU\delta \;\Rightarrow\; h = \frac{2}{3}\delta \qquad \boxed{+4}$$

(b) Linear Momentum Equation in x Direction:

$$\sum F_x = -\int_1 u(y)\rho[u(y)]\, b\, dy + \int_2 u(y)\rho[u(y)]\, b\, dy \qquad \boxed{+2}$$

$$-F = -\int_0^h \rho U^2 b\, dy + \int_0^\delta \rho u^2(y)\, b\, dy$$

$$-F = -\rho U^2 bh + \rho b \int_0^\delta u^2(y) dy$$

$$= -\rho U^2 bh + \rho b U^2 \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^2 dy$$

$$\int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)^2 dy = \int_0^\delta \left(\frac{4y^2}{\delta^2} + \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3}\right) dy = \left(\frac{4y^3}{3\delta^2} + \frac{y^5}{5\delta^4} - \frac{y^4}{\delta^3}\right)_0^\delta = \frac{4}{3}\delta + \frac{1}{5}\delta - \delta = \frac{8}{15}\delta$$

$$-F = -\rho U^2 bh + \rho b U^2\left(\frac{8}{15}\delta\right)$$

$$= -\rho U^2 b\left(\frac{2}{3}\delta\right) + \rho b U^2\left(\frac{8}{15}\delta\right)$$

$$\therefore\; F = \frac{2}{15}\rho U^2 b\delta \qquad \boxed{+2}$$

Solution 2:

Assumptions:

(1) Steady: $\frac{\partial}{\partial t} = 0$
(2) 2D flow: $w = 0$; $\frac{\partial}{\partial z} = 0$
(3) No gravity: $g = 0$
(4) Pressure constant: $\partial p/\partial x_i = 0$
(5) Vertical velocity constant everywhere: $v = -v_0$; $\partial v/\partial x_i = 0$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} + 0(5) + 0(2) = 0$$

$$\frac{\partial u}{\partial x} = 0 \quad (6)$$

+3

Navier-Stokes in *x* direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(0(1) + 0(6) + (-v_0)\frac{\partial u}{\partial y}(5) + 0(2)\right) = 0(3) - 0(4) + \mu\left(0(6) + \frac{\partial^2 u}{\partial y^2} + 0(2)\right)$$

$$\mu\frac{\partial^2 u}{\partial y^2} + \rho v_0\frac{\partial u}{\partial y} = 0$$

+4

Follow the hint to solve the O.D.E.:

$$u = e^{\lambda y} \quad \rightarrow \quad \frac{\partial u}{\partial y} = \lambda e^{\lambda y}; \ \frac{\partial^2 u}{\partial y^2} = \lambda^2 e^{\lambda y}$$

Replace in the differential equation and find λ values:

$$\mu\lambda^2 e^{\lambda y} + \rho v_0 \lambda e^{\lambda y} = 0$$

$$e^{\lambda y}(\mu\lambda^2 + \rho v_0 \lambda) = 0 \quad \rightarrow \quad \mu\lambda^2 + \rho v_0\lambda = 0 \quad \rightarrow \quad \lambda(\mu\lambda + \rho v_0) = 0 \quad \rightarrow \quad \lambda = \begin{cases} 0 = \lambda_1 \\ -\dfrac{\rho v_0}{\mu} = \lambda_2 \end{cases}$$ +1

Therefore:

$$u(y) = C_1 e^{\lambda_1 y} + C_2 e^{\lambda_2 y}$$

$$u(y) = C_1 e^{(0)y} + C_2 e^{\left(-\frac{\rho v_0}{\mu}\right)y}$$

$$u(y) = C_1 + C_2 e^{-\frac{\rho v_0}{\mu}y}$$ +1

Boundary conditions:

$$u = 0 \; at \; y = 0 \quad \rightarrow \quad 0 = C_1 + C_2$$

$$u = U \; at \; y = \infty \quad \rightarrow \quad U = C_1 + 0$$

Find C_1 and C_2:

$$C_1 = U\,; \quad C_1 = -U$$ +1

Replace and find final solution:

$$u(y) = U - Ue^{-\frac{\rho v_0}{\mu}y} = U\left(1 - e^{-\frac{\rho v_0}{\mu}y}\right)$$

Solution 3:

(a)

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 A} \quad \textbf{(1)}$$

$$A = \frac{\pi D^2}{4} \quad \textbf{(1)}$$

If in equilibrium at constant velocity, then:

$$D = W \quad \textbf{(1)}$$

$$C_D = \frac{W}{\frac{1}{2}\rho V^2 A} = \frac{(200)}{\frac{1}{2}(1.2)(3)^2 \frac{\pi}{4}(5)^2} = 1.89 \quad \textbf{(1)}$$

(b)

To repeat the experiment, it has to be designed to reach same Reynolds number as prototype:

$$Re_m = Re_p \quad \textbf{(2)}$$

$$\frac{V_m D_m}{\nu_m} = \frac{V_p D_p}{\nu_p} \quad \rightarrow \quad V_m = V_p \frac{D_p}{D_m} = V_p \lambda = (3)(2.5) = 7.5\ m/s \quad \textbf{(1)}$$

At this speed the drag coefficient should be the same as prototype since $C_D = f(Re)$. Therefore:

$$C_{D_m} = C_{D_p} = 1.89 \quad \textbf{(2)}$$

$$W_{total} = C_D \frac{1}{2}\rho V^2 A = (1.89)(0.5)(1.2)(7.5)^2 \frac{\pi}{4}(5/2.5)^2 = 200\ N$$

$$W_{load} = W_{total} - W_{Parachute} = 200 - 40 = 160 \quad \textbf{(1)}$$

Solution 4:

(a)

$$\tau_w = \rho \frac{U^2}{2} c_f \quad (0.5)$$

$$c_f \approx \frac{0.027}{Re_x^{1/7}} \quad (0.5)$$

$$\tau_w = \rho \frac{U^2}{2} \frac{0.027}{Re_x^{1/7}} \quad (1)$$

$$= \rho \frac{U^2}{2} \frac{0.027}{\left(\frac{\rho U x}{\mu}\right)^{1/7}} \quad (1)$$

$$= \frac{0.0135 \mu^{1/7} \rho^{6/7} U^{13/7}}{x^{1/7}}$$

Re-arrange to find U:

$$U = \left(\frac{\tau_w \, x^{1/7}}{0.0135 \mu^{1/7} \rho^{6/7}}\right)^{7/13}$$

(1)

$$= \left(\frac{(1.0)(0.4)^{1/7}}{0.0135\,(1.8E-5)^{1/7}(1.2)^{6/7}}\right)^{7/13} = 20.16\, m/s \quad (0.5)$$

(b)

$$(1) \quad Re_x = \frac{\rho U x}{\mu} = \frac{(1.2)(20.16)(0.4)}{(1.8E-5)} = 537{,}600 > 5 \times 10^5 \quad OK, turbulent \quad (1)$$

(c)

$$C_D \approx \frac{0.031}{Re_L^{1/7}} \quad (1)$$

$$(1) \quad Re_L = \frac{\rho U L}{\mu} = \frac{(1.2)(20.16)(1.0)}{(1.8E-5)} = 1{,}344{,}000$$

$$C_D = \frac{0.031}{(1344000)^{1/7}} = 0.00413$$

$$D = C_D \frac{1}{2} \rho U^2 b L \quad (1)$$

$$= (0.00413)(0.5)(1.2)(20.16)^2(0.7)(1.0) = 0.7\, N \quad (0.5)$$

Solution 5:

a) Continuity:

$$Q_1 + Q_2 = \frac{\pi}{4}d_1^2 V_1 + \frac{\pi}{4}d_2^2 V_2 = Q_{total}; \quad V_2 = \frac{4}{\pi d_2^2}Q_{total} - \frac{d_1^2}{d_2^2}V_1 \quad \textbf{(1)}$$

$$V_2 = \frac{4}{\pi 0.04^2}0.036 - \frac{0.05^2}{0.04^2}V_1 \quad \textbf{(1)}$$

$$V_2 = 28.65 - 1.56\,V_1$$

(b)

Same head loss for parallel pipes:

$$h_{f1} = h_{f2} + h_{m2} \quad \textbf{(2)}$$

$$f_1 \frac{L_1}{d_1}\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\left(f_2 \frac{L_2}{d_2} + K\right) = 0$$

$$f_1 \frac{60}{0.05}\frac{V_1^2}{2 \times 9.81} - \frac{V_2^2}{2 \times 9.81}\left(f_2 \frac{55}{0.04} + 1.5\right) = 0 \quad \textbf{(1)}$$

$$61.16 f_1 V_1^2 - (28.65 - 1.56\,V_1)^2(70.08 f_2 + 0.076) = 0 \quad \textbf{(1)}$$

Reynolds Number:

$$Re_1 = \frac{\rho V_1 D_1}{\mu} = \frac{998 \times 0.05}{0.001}V_1 = 49900\,V_1 \quad \textbf{(0.5)}$$

$$Re_2 = \frac{\rho V_1 D_1}{\mu} = \frac{998 \times 0.04}{0.001}V_2 = 39920 V_2 \quad \textbf{(0.5)}$$

Relative roughness:

$$\frac{\epsilon}{D_1} = \frac{0.15}{50} = 0.003 \quad \textbf{(0.5)}$$

$$\frac{\epsilon}{D_2} = \frac{0.15}{40} = 0.00375 \quad \textbf{(0.5)}$$

Guessing $f_1 = f_2 = 0.02$

$$f_1 = 0.02, f_2 = 0.02 \rightarrow V_1 = 11.59 \rightarrow V_2 = 10.54 \rightarrow Re_1 = 57800, Re_2 = 421000 \quad \textbf{(1)}$$

$$f_1 = 0.0264, f_2 = 0.0282 \rightarrow V_1 = 11.69 \rightarrow V_2 = 10.37$$

$$\therefore V_1 = 11.69\,m/s \quad \textbf{(1)}$$

Solution 6

a) Velocity

$$\theta = \frac{\pi}{2} = 90^0$$

$$a = \frac{4}{\pi} = 1.27$$

$$r = \frac{m(\pi - \theta)}{U \sin\theta} = \frac{a(\pi - \theta)}{\sin\theta} = \frac{\pi}{2} a$$

+2

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{1}{r}(U \text{ r} \cos\theta + m) = U\cos\theta + \frac{m}{r}$$

$$v_\theta = -\frac{\partial \Psi}{\partial r} = -U\sin\theta$$

$$V^2 = U^2 \sin^2\theta + U^2\cos^2\theta + \frac{m^2}{r^2} + \frac{2Um}{r}\cos\theta = U^2 + \frac{U^2 a^2}{r^2} + \frac{2U^2 a}{r}\cos\theta$$

$$V^2 = U^2\left(1 + \frac{a^2}{r^2} + \frac{2a}{r}\cos\theta\right)$$

$$V_A^2 = U^2\left(1 + \frac{a^2}{\pi^2 a^2/4} + \frac{2a}{\pi a/2}\cos\frac{\pi}{2}\right) = 1.405U^2$$

$$V_A = 1.185U = 1.185 \times 2.5 = 2.96\, m/s$$

+4

b) Bernoulli

$$\frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B$$

$$p_A = p_B + \frac{\gamma}{2g}(V_B^2 - V_A^2) + \gamma(z_B - z_A)$$

$$p_A = \frac{130000}{9790} + \frac{9790}{2 \times 9.81}(2.5^2 - 2.96^2) + 9790(-2) = 109200\, Pa$$

+4