# ME:5160 (58:160) Intermediate Mechanics of Fluids

# Fall 2021 – HW6 Solution

**P4.2** Flow through the converging nozzle in Fig. P4.2 can be approximated by the one-dimensional velocity distribution

$$u \approx V_o\left(1+\frac{2x}{L}\right) \quad \upsilon \approx 0 \quad w \approx 0$$

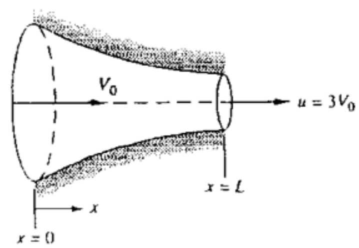



**Fig. P4.2**

(a) Find a general expression for the fluid acceleration in the nozzle. (b) For the specific case *V*o = 10 ft/s and *L* = 6 in, compute the acceleration, in *g*'s, at the entrance and at the exit.

**Solution:** Here we have only the single 'one-dimensional' convective acceleration:

$$\frac{du}{dt} = u\frac{\partial u}{\partial x} = \left[V_o\left(1+\frac{2x}{L}\right)\right]\frac{2V_o}{L} = \mathbf{\frac{2V_o^2}{L}\left(1+\frac{2x}{L}\right)} \quad \textit{Ans. } \text{(a)}$$

$$\textit{For } L = 6'' \textit{ and } V_o = 10\ \frac{ft}{s},\quad \frac{du}{dt} = \frac{2(10)^2}{6/12}\left(1+\frac{2x}{6/12}\right) = 400(1+4x),\ \textit{with x in feet}$$

At x = 0, du/dt = **400** ft/s$^2$ (12 g's); at x = L = 0.5 ft, du/dt = **1200** ft/s$^2$ (37 g's). *Ans.* (b)

***P4.79** Study the combined effect of the two viscous flows in Fig. 4.12. That is, find $u(y)$ when the upper plate moves at speed $V$ and there is also a constant pressure gradient ($dp/dx$). Is superposition possible? If so, explain why. Plot representative velocity profiles for (a) zero, (b) positive, and (c) negative pressure gradients for the same upper-wall speed $V$.

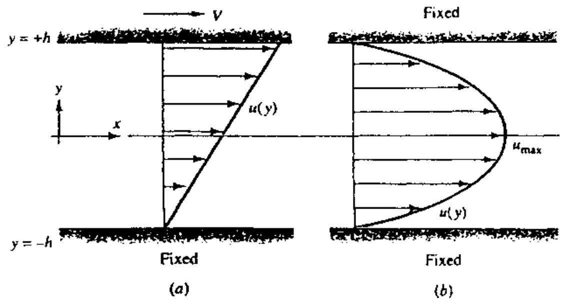



**Fig. 4.16**

**Solution:**

Continuity

Fully developed

2D

$$\cancel{\frac{\partial u}{\partial x}}+\cancel{\frac{\partial v}{\partial y}}+\cancel{\frac{\partial w}{\partial z}}=0$$

Steady

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}+u\cancel{\frac{\partial u}{\partial x}}+v\cancel{\frac{\partial u}{\partial y}}+\cancel{w}\frac{\partial u}{\partial z}\right)=\rho\cancel{g_x}-\frac{\partial p}{\partial x}+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}+\cancel{\frac{\partial^2 u}{\partial z^2}}\right)=$$

2D    Neglect gravity    2D

$$\therefore 0=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\Rightarrow \mu\frac{d^2u}{dy^2}=\frac{\partial p}{\partial x}$$

Integrate twice:

$$u=\left(\frac{1}{\mu}\frac{\partial p}{\partial x}\right)\frac{y^2}{2}+C_1y+C_2$$

Boundary conditions:

at $y$ = -$h$: $u(-h)=0$

at $y$ = +$h$: $u(+h)=V$

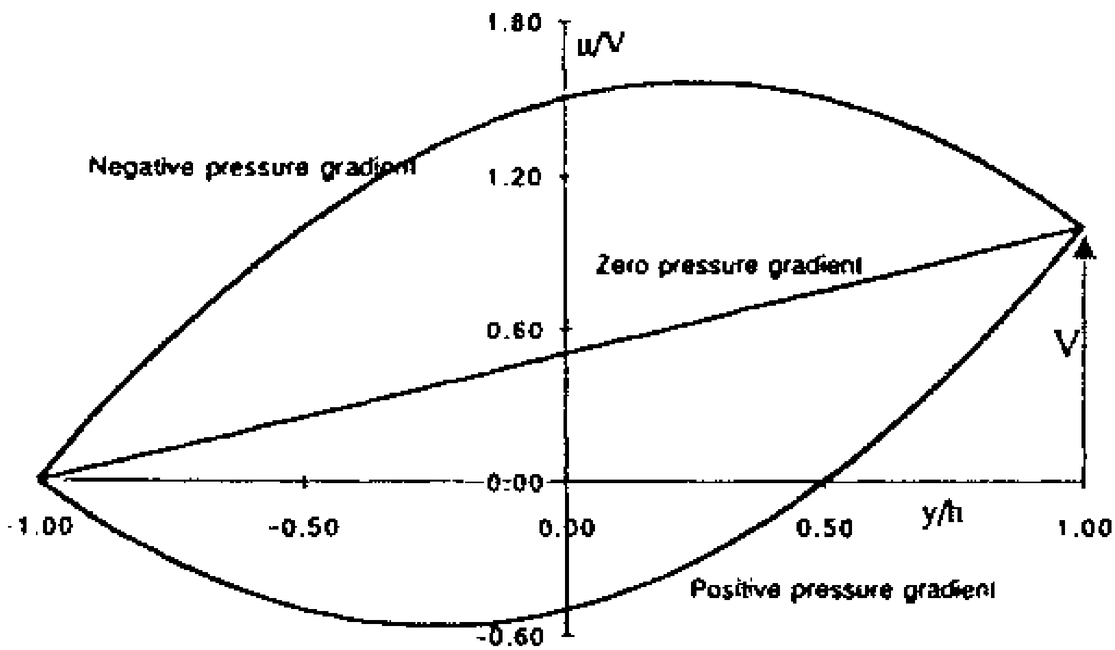
$$\therefore \quad C_1=\frac{V}{2h},\quad C_2=\frac{V}{2}-\left(\frac{1}{\mu}\frac{\partial p}{\partial x}\right)\frac{h^2}{2}$$

Combined solution is

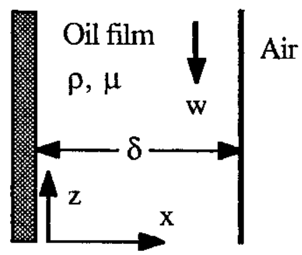
$$\therefore \quad u = \left(\frac{1}{\mu}\frac{\partial p}{\partial x}\right)\frac{y^2}{2} + \frac{V}{2h}y + \frac{V}{2} - \left(\frac{1}{\mu}\frac{\partial p}{\partial x}\right)\frac{h^2}{2}$$

$$u = \frac{V}{2}\left(1+\frac{y}{h}\right) + \frac{h^2}{2\mu}\left(-\frac{dp}{dx}\right)\left(1-\frac{y^2}{h^2}\right)$$

The superposition is quite valid because the convective acceleration is zero, hence what remains is linear: $\nabla p = \mu\nabla^2\mathbf{V}$. Three representative velocity profiles are plotted at right for various (dp/dx).



***P4.80** An oil film drains steadily down the side of a vertical wall, as shown. After an initial development at the top of the wall, the film becomes independent of z and of constant thickness. Assume that w = w(x) only that the atmosphere offers no shear resistance to the film. (a) Solve Navier-Stokes for w(x). (b) Suppose that film thickness and $[\partial w/\partial x]$ at the wall are measured. Find an expression which relates $\mu$ to this slope $[\partial w/\partial x]$.




**Solution:** First, there is no pressure gradient $\partial p/\partial z$ because of the constant-pressure atmosphere. The Navier-Stokes z-component is $\mu(d^2w/dx^2) = \rho g$, and the solution requires w = 0 at x = 0 and (dw/dx) = 0 (no shear at the film edge) at x = $\delta$. The solution is:

$$w = \frac{\rho g x}{2\mu}(x - 2\delta) \quad \textit{Ans.}\text{ (a)} \quad \text{NOTE: } w \text{ is negative (down)}$$

The wall slope is $dw/dx|_{wall} = -\rho g\delta/\mu$, rearrange: $\mu = -\rho g\delta/[dw/dx|_{wall}]$ *Ans.* (b)

**P4.88** The viscous oil in Fig. P4.88 is set into steady motion by a concentric inner cylinder moving axially at velocity *U* inside a fixed outer cylinder. Assuming constant pressure and density and a purely axial fluid motion, solve Eqs. (4.38) for the fluid velocity distribution *vz*(*r*). What are the proper boundary conditions?

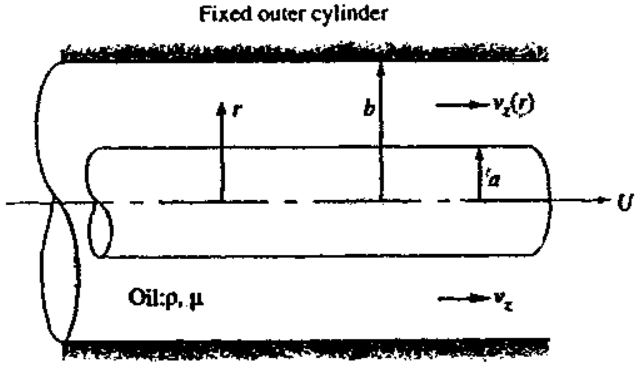



Fig. P4.88

**Solution:** If vz = fcn(r) only, the *z*-momentum equation (Appendix E) reduces to:

$$\rho\frac{dv_z}{dt} = -\frac{\partial p}{\partial z} + \rho g_z + \mu\nabla^2 v_z, \quad \text{or:} \quad 0 = 0 + 0 + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

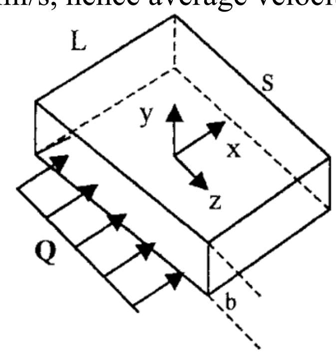
The solution is vz = C1 ln(r) + C2, subject to vz(a) = U and vz(b) = 0

Solve for C1 = U/ln(a/b) and C2 = –C1 ln(b)

The final solution is: $$\mathbf{v_z = U\frac{\ln(r/b)}{\ln(a/b)}}$$ *Ans.*

**C4.1** In a certain medical application, water at room temperature and pressure flows through a rectangular channel of length $L$ = 10 cm, width $s$ = 1 cm, and gap thickness $b$ = 0.3 mm. The volume flow is sinusoidal, with amplitude $Q_o$ = 0.5 ml/s and frequency $f$ = 20 Hz, that is, $Q = Q_o\sin(2\pi f\, t)$.
(a) Calculate the maximum Reynolds number Re = $Vb/\nu$, based on maximum average velocity and gap thickness. Channel flow remains *laminar* for Re < 2000, otherwise it will be *turbulent*. Is this flow laminar or turbulent?
(b) Assume quasi-steady flow, that is, solve as if the flow were steady at any given Q(t). Find an expression for streamwise velocity $u$ as a function of $y$, $\mu$, dp/dx, and $b$, where dp/dx is the pressure gradient required to drive the flow through the channel at flow rate Q. Also estimate the maximum magnitude of velocity component $u$.
(c) Find an analytic expression for flow rate Q(t) as a function of dp/dx.
(d) Estimate the wall shear stress $\tau_w$ as a function of Q, f, $\mu$, $b$, $s$, and time $t$.
(e) Finally, use the given numbers to estimate the wall shear amplitude, $\tau_{wo}$, in Pa.

**Solution:** (a) Maximum flow rate is the amplitude, $Q_o$ = 0.5 ml/s, hence average velocity V = Q/A:




$$V = \frac{Q}{bs} = \frac{0.5E\text{–}6\ m^3/s}{(0.0003\,m)(0.01\,m)} = 0.167\,m/s$$

$$\mathrm{Re}_{\max} = \frac{Vb}{\nu} = \frac{(0.167)(0.0003)}{(0.001/998)} = \mathbf{50\ (laminar)} \quad \textit{Ans.}\ \text{(a)}$$

(b, c) The quasi-steady analysis is just like Eqs. (4.137-4.139) of the text, with "h" = b/2:

$$\boldsymbol{u = \frac{-1}{2\mu}\frac{dp}{dx}\left(\frac{b^2}{4} - y^2\right),\quad u_{\max} = \frac{-1}{2\mu}\frac{dp}{dx}\frac{b^2}{4},\quad Q_{\max} = \frac{2}{3}u_{\max}bs = \frac{-sb^3}{12\mu}\frac{dp}{dx}} \quad \textit{Ans.}\ \text{(b, c)}$$

$$\text{(d) Wall shear:}\quad \tau_{wall} = \mu\left|\frac{du}{dy}\right|_{wall} = \boldsymbol{\frac{b}{2}\frac{dp}{dx} = \frac{6\mu Q}{sb^2} = \frac{6\mu Q_o}{sb^2}\sin(2\pi f\, t)} \quad \textit{Ans.}\ \text{(d)}$$

(e) For our given numerical values, the amplitude of wall shear stress is:

$$\tau_{wo} = \frac{6\mu Q_o}{sb^2} = \frac{6(0.001)(0.5E\text{–}6)}{(0.01)(0.0003)^2} = \boldsymbol{3.3\ Pa} \quad \textit{Ans.}\ \text{(e)}$$