Problem 3

For the given load,

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which can be used to obtain,

$$W = \frac{\sigma^2}{2E}.$$

From Hooke's law,

$$\boldsymbol{\varepsilon} = \begin{bmatrix} -\nu\sigma/E & 0 & 0 \\ 0 & \sigma/E & 0 \\ 0 & 0 & -\nu\sigma/E \end{bmatrix},$$

which can be integrated to obtain

$$\boldsymbol{u} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} -\nu\sigma x/E \\ \sigma y/E \\ -\nu\sigma z/E \end{Bmatrix} \text{ and } \frac{\partial \boldsymbol{u}}{\partial x} = \begin{Bmatrix} \partial u_x/\partial x \\ \partial u_y/\partial x \\ \partial u_z/\partial x \end{Bmatrix} = \begin{Bmatrix} -\nu\sigma/E \\ 0 \\ 0 \end{Bmatrix}.$$

Also,

$$\boldsymbol{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix}$$

and

$$y = R\sin\theta; \quad dy = R\cos\theta d\theta; \quad d\Gamma = Rd\theta.$$

Hence,

$$\boldsymbol{T} = \boldsymbol{\sigma n} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \sigma\sin\theta \\ 0 \end{Bmatrix}$$

and

$$\boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} = \boldsymbol{T}^T \frac{\partial \boldsymbol{u}}{\partial x} = \{0 \quad \sigma\sin\theta \quad 0\} \begin{Bmatrix} -\nu\sigma/E \\ 0 \\ 0 \end{Bmatrix} = 0.$$

Also,

$$\int_\Gamma W dy = \int_0^{2\pi} \frac{\sigma^2}{2E} R\cos\theta d\theta = \frac{\sigma^2}{2E} R\left[\sin\theta\right]_0^{2\pi} = 0$$

Therefore,

$$\boxed{J = \int_\Gamma \left[W dy - \boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} d\Gamma \right] = \int_\Gamma W dy - \int_\Gamma \boldsymbol{T} \cdot \frac{\partial \boldsymbol{u}}{\partial x} d\Gamma = 0 - 0 = 0}.$$