Problem 2:

From Problem 1,

$$\begin{aligned} \sigma_1 &= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\right] \\ \sigma_2 &= \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\right] \\ \sigma_3 &= \begin{cases} 0, & \text{plane stress} \\ \nu(\sigma_1+\sigma_2), & \text{plane strain} \end{cases} \end{aligned} \quad . \tag{10}$$

(a) Expressions of Plastic Zone Boundary (Mode-I; Tresca)

<u>Plane Stress</u>

The maximum shear stress τ_{max} is (Note, $\sigma_1 \geq \sigma_2 \geq \sigma_3$ for $0 \leq \theta \leq \pi$)

$$\tau_{max} = \frac{\sigma_1-\sigma_3}{2} = \frac{1}{2}\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\right]. \tag{11}$$

Hence, the Tresca yield criterion becomes

$$\tau_{max}\big|_{r=r_y} = \frac{S_y}{2}, \tag{12}$$

i.e.,

$$\frac{1}{2}\frac{K_I}{\sqrt{2\pi r_y}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\right] = \frac{S_y}{2}, \tag{13}$$

which can be solved to obtain

$$\boxed{r_y(\theta) = \frac{K_I^{\,2}}{2\pi S_y^{\,2}}\cos^2\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\right]^2, \quad 0 \leq |\theta| \leq \pi.} \tag{14}$$

Note, Equation 14 is also valid for $-\pi \leq \theta \leq 0$, as the plastic zone is symmetrical w.r.t. crack (horizontal axis) line.

Plane Strain

To obtain τ_{max}, ordering of principal stresses is important. For $0 \le \theta \le \pi$ and $0 \le \nu \le 0.5$, σ_1 is always the largest. However, for σ_2 and σ_3, there are two possibilities, i.e.,

$$\sigma_1 \ge \sigma_2 \ge \sigma_3 \quad \text{or} \quad \sigma_1 \ge \sigma_3 \ge \sigma_2,$$

depending on the value of the Poisson's ratio. Hence,

$$\tau_{max} = \begin{cases} \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{\sigma_1(1-\nu) - \nu\sigma_2}{2}, & \text{if } \sigma_1 \ge \sigma_2 \ge \sigma_3 \\ \dfrac{\sigma_1 - \sigma_2}{2}, & \text{if } \sigma_1 \ge \sigma_3 \ge \sigma_2 \end{cases}. \tag{15}$$

The Tresca yield criterion is

$$\tau_{max} = \begin{cases} \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{\sigma_1(1-\nu) - \nu\sigma_2}{2}, & \text{if } \sigma_1 \ge \sigma_2 \ge \sigma_3 \\ \dfrac{\sigma_1 - \sigma_2}{2}, & \text{if } \sigma_1 \ge \sigma_3 \ge \sigma_2 \end{cases} = \frac{S_y}{2}. \tag{16}$$

which can be solved to obtain

$$r_y(\theta) = \begin{cases} \dfrac{K_I^{\,2}}{2\pi S_y^{\,2}} \cos^2\dfrac{\theta}{2}\left[(1-2\nu) + \sin\dfrac{\theta}{2}\right]^2, & \text{if } \sigma_1 \ge \sigma_2 \ge \sigma_3 \\ \dfrac{K_I^{\,2}}{2\pi S_y^{\,2}} \sin^2\theta, & \text{if } \sigma_1 \ge \sigma_3 \ge \sigma_2 \end{cases}. \tag{17}$$

or,

$$\boxed{r_y(\theta) = \frac{K_I^{\,2}}{\pi S_y^{\,2}} \times \frac{1}{2} \max\left\{\cos^2\frac{\theta}{2}\left[(1-2\nu) + \sin\frac{\theta}{2}\right]^2, \sin^2\theta\right\}, \quad 0 \le |\theta| \le \pi}. \tag{18}$$

Note, Equation 14 is also valid for $-\pi \le \theta \le 0$, as the plastic zone is symmetrical w.r.t. crack (horizontal axis) line.