1-9 PROPERTIES OF CONSERVATIVE SYSTEMS

- *Law of Conservation of Mechanical Energy*: If system travels from configuration X_1 to X, the work W′ equals the change in k.e., i.e., $W' = \Delta T$. If system is conservative in the kinetic sense, W′ also equals the loss of potential, $W' = -\Delta V$.

  (1-11) $\Delta T + \Delta V = 0$, or $T + V = \text{constant}$

Principle of Stationary Potential Energy

- For a conservative system, the principle of virtual work can be expressed as $\lim_{s\to 0} (\Delta V/s) = 0$. It is frequently possible to write ΔV in the variational form as

  (1-12) $\Delta V = \delta V + 1/2\, \delta^2 V + o(s^3)$

  The <u>Principle of virtual work is equivalent to $\delta V = 0$</u>. This is called the *principle of stationary potential energy*.

- *Equilibrium condition*: for any stationary system with finite D.O.F., (1-9) with the equilibrium condition $Q_i = 0$, gives

  (1-13) $P_i = U_{,i}$

- <u>Sufficient condition for equilibrium of a system</u>.
- <u>Necessary and sufficient condition for a holonomic and unchecked system</u>.

- For elastic structures:

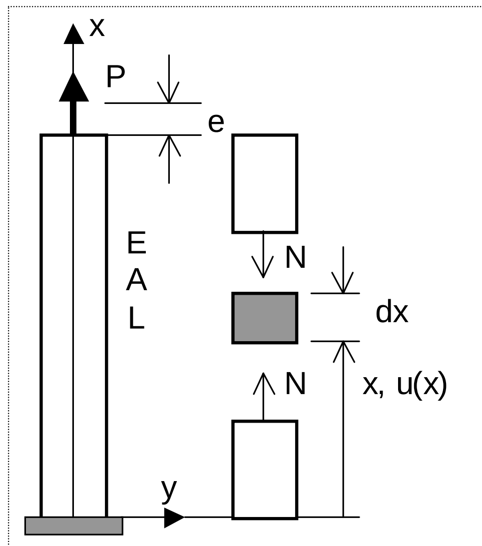
  (1-14) $U = 1/2\, a_{ij}\, x_i x_j$

- Principle of stationary p.e. gives

  (1-15) $\mathbf{Ax = P}$

EXAMPLE: POTENTIAL AND STRAIN ENERGIES OF UNIFORM ROD SUBJECTED TO A CONSTANT LOAD

- The figure shows the rod. Let e be the extension of the rod, which will be treated as the g.c. in the inertial reference frame x-y.



- u(x) be the displacement of the rod at the distance x. Since the rod is uniform, u(x) = ex/L.
- *Potential energy of the external load*:

  The virtual work of the external load P in going through the virtual displacement de is an indefinite integral given as

$$W_e = \int P\, de$$

Therefore the potential energy of the external load is

$$\Omega = -W_e = -\int P\, de = -Pe$$

- *Strain energy*:

To calculate virtual work of the internal force, consider a differential element dx at a distance x as shown in the figure. The V.W. done by the internal forces on the differential element is given as the following indefinite integral:

$$dW_i = \int \left[N(x)du(x) - N(x+dx)du(x+dx) \right]$$

Note that N(x) = N(x+dx) = N. Substitute for u(x) = ex/L.

$$dW_i = \int \left[N \frac{xde}{L} - N \frac{(x+dx)de}{L} \right]$$

Simplifying the equation, we get

$$dW_i = -\int \left[N \frac{dxde}{L} \right]$$

Noting that N = EAe/L, the above expression gives the V.W. done on the differential element as

$$dW_i = -\frac{EAe^2}{2L^2} dx$$

Integrating the expression over the length L, we get the V.W. of the internal forces for the system as

$$W_i = -\int_0^L \left(\frac{EAe^2}{2L^2} \right) dx = -\frac{EAe^2}{2L} = -\tfrac{1}{2} Ne$$

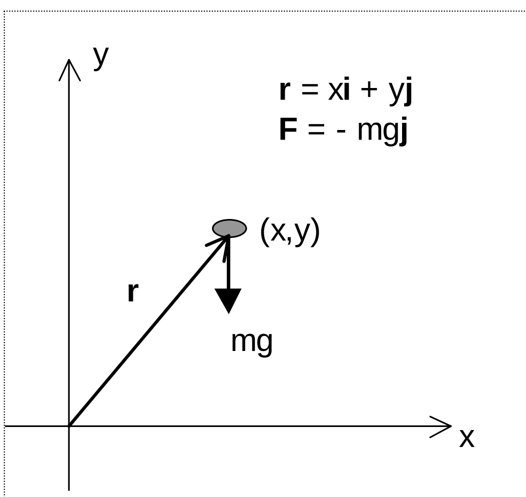
Therefore the strain energy is given as

$$U = \frac{EAe^2}{2L} = \tfrac{1}{2} Ne$$

EXAMPLE:

POTENTIAL ENERGY OF GRAVITATIONAL FORCE

- Consider a particle in the gravitational field as shown in the figure. V.W. of force mg is given as the indefinite integral:



$$W_e = \int \boldsymbol{F} \bullet \boldsymbol{dr} = -\int mgdy = -mgy$$

Therefore the potential energy is

$$\boldsymbol{\Omega} = -\boldsymbol{W_e} = \boldsymbol{mgy}$$

1-10 POTENTIAL ENERGY OF A SYSTEM OF PARTICLES

- *Potential energy* of two and n particles systems

  For two particles system (F is force of attraction).

  dW = **F**•**dr** = -Fdr; dV = -dW

  (1-16) $$V = \int F dr$$

  For an n-particle system, there are n(n-1)/2 *bonds*. This is also a conservative system. The total potential energy V is the summation of p.e. of each of the bonds.

  $$V = \sum_{i=1}^{M} V_i ; \qquad V_i = \int F_i dr_i ; \qquad M=n(n-1)/2$$

Force Field

- Let us consider a system composed of one movable particle and several fixed particles. The force **F** which acts on the movable particle due to its bond between the fixed particle, is a vector point function or a vector field. The potential energy V of this system is a function of the position of the movable particle x_i:

$$dV = \frac{\partial V}{\partial x_i} dx_i$$

$$dV = -dW = -F_i dx_i$$

(1-18) $\mathbf{F} = -\text{grad } V$

- A *force field* that possesses a *potential energy function* is said to be <u>conservative</u>.
- If x_i are rectangular coordinates, and F_i are the components of the vector **F**, an infinitesimal vector d**x** in the direction of **F** conforms to the differential relation:

(1-17) $\frac{dx_1}{F_1} = \frac{dx_2}{F_2} = \frac{dx_3}{F_3}$; or $\frac{dx}{F_x} = \frac{dy}{F_y} = \frac{dz}{F_z}$

- The general integral of these equations represents a system of curves called "*lines of force*", or *flow lines*.

Newtonian Potential

- <u>Newton's law of gravitation</u>: the force of attraction between any two point masses is proportional to inverse of the square of the distance between the particles

(1-19) $F = k\frac{mm'}{r^2}$

- The potential energy of this system using (1-16), is

(1-20) $V = -k\frac{mm'}{r}$

- Let k = 1, m' = 1. Then potential energy of a unit mass in a force field that is generated by n fixed masses m_i is

  (1-21) $$V = -\sum \frac{m_i}{r_i}$$

  r_i = distance between the ith fixed mass and the movable mass.

- The potential energy of a unit mass in a force field that is generated by a *continuous fixed distributed mass* with variable density ρ:

  (1-22) $$V = -\int \frac{\rho \, ds}{r}$$

  where r is the distance from a particle of the distributed mass to the movable particle. It could be a line integral, surface integral or a volume integral depending on the distributed mass.

- Newtonian potential functions (Eqs. 1-21 and 1-22) are solutions of Laplace's differential equation $\nabla^2 V=0$. Any function that satisfies Laplace's equation is called a *Harmonic* function.

- Example of distributed electric charge.