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STABILITY CRITERIA: A REVIEW

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Abstract—The basic theories of stability of mechanical and structural systems are described. The theory for discrete dynamical systems that has applications in control of systems is reviewed first. The variational stability criteria for nonlinear structural systems are then developed for both the conservative and nonconservative systems. For conservative systems, the existence of minimum potential energy is combined with the total Lagrangian formulation to construct the stability criterion. The Hamilton's principle and the concept of adjacent states of motion are adopted to investigate the criterion for nonconservative systems. To study computational aspects, two beam–column examples are presented: one with conservative force and the other with nonconservative force. The examples provide insights for numerical implementation of the criteria.

1. INTRODUCTION

Long after Euler's classic investigation of an axially compressed column in 1744, Poincaré [1] presented fundamentals for a general theory of elastic stability. He showed that a loss of stability is normally associated with either a limit point, which represents a local extremum on an initial path, or a bifurcation, which is the intersection of the initial equilibrium path with the branch path. Liapunov [2, 3] also investigated the stability of motion of systems using first order differential equations. His theories, especially the Liapunov direct method (or, so-called second method) have been widely used by several researchers [4–10]. In the context of an elastic continuum, Bryan [11] appears to be the first researcher who attempted to develop a general theory of stability. His analysis is based on the energy criterion as a postulate generalized directly from the well-known Lagrange theorem for discrete mechanical systems, and it is believed that this is the first adaptation of the extremum properties of the potential energy to continuous systems. Later on, researchers such as Southwell [12], Biezeno and Hencky [13], Reissner [14], Trefftz [15, 16], Marguerre [17], Kappus [18] and Biot [19, 20] limited their studies to determination of the stability limit rather than examining the behavior of the system on reaching and exceeding this limit. Trefftz [16] presented a variational principle that yielded the critical load. Folowing that, Koiter [21–27] gave a systematic nonlinear theory of stability in his classical thesis and his subsequent papers. His work has contributed significantly to the understanding of nonlinear elastic stability.

Thompson and Hunt [28, 29] and Huseyin [30, 31] give a complete summary of the theory of nonlinear elastic stability in their publications. Though, they only emphasize discrete systems, the theories can be extended to continua. Huseyin [30, 31] also discusses the theory of nonlinear stability with multiple loading parameters. The analysis is suitable for those systems with multi-pattern loading. Thompson and Hunt [29] have published a collection of all the papers presented at the International Union of Theoretical and Applied Mechanics Symposium in 1982. Both the theoretical developments and experimental results are summarized.

As for systems with nonconservative forces, Bolotin [32] gave fundamental approach to the determination of the stability limit. Plaut [33, 34] studied the stability limit of nonself-adjoint linear partial differential equations with time-varying coefficients. His model can be used to represent some linear continuous systems, i.e. elastic columns and plates, and cantilever beams subjected to random follower forces. His subsequent papers [35–37], however, concentrated on the post-buckling behavior of discrete systems. The imperfection-sensitivity was studied by the perturbation techniques. Leipholz [38] presented the application of energy methods to the stability of nonconservative systems. Analyses for single- and multi-degree-of-freedom systems were presented. Papastavridis [39–43] used the Hamilton principle to analyze the stability of motion of discrete systems using the concept of adjoint configuration.

The stability investigation of conservative systems is well developed and documented. Systematic procedures have been developed for both discrete and continuum systems. On the other hand, stability of nonconservative systems has been explored only relatively recently. Although, the fundamentals of investigating the critical point are well established, a unified criterion for a continuum is not yet available.

Besides reviewing the classical theories for both discrete and continuous systems with conservative forces, this paper presents a variational form of the stability criterion for nonconservative systems. Geometric as well as material nonlinearities are included. In addition, a general condition is derived in the variational form that can distinguish between limit point and bifurcation point instability. Two numerical examples are also solved which give valuable insights into the stability of nonconservative systems. The motivation for the present review is to study stability criteria for nonlinear systems, so that constraints on the critical load can be included in the optimum design formulation. It has been shown previously [44] that such constraints must be included in the optimum design process for complex systems; otherwise the process fails and an optimum design of the system cannot be obtained.

It is noted that the literature on applications of the stability theory to particular structural and mechanical systems is quite extensive. That literature is not included in this review. Only the literature that contains fundamental concepts and basic theories of stability is cited.

2. STABILITY OF DISCRETE DYNAMICAL SYSTEMS

2.1. *Definitions of stability*

Let the motion at time t of a point of the system in n dimensional space ${}^t x_i(t)$, $i = 1 \ldots n$, continuously depend on certain parameters a_j, $j = 1 \ldots m$, which also include initial condition at time t_0. The motion of the system can be written as functions of a_j as ${}^t x_i(a_j)$. Let a_j be unperturbed parameters corresponding to unperturbed motion ${}^t x_i(a_j)$ and $\bar{a}_j$ be the perturbed parameters in the neighborhood of a_j corresponding to the perturbed motion ${}^t \bar{x}_i(\bar{a}_j)$, where the overbar indicates the perturbed state. The stability definition can be introduced according to Liapunov [2] as follows.

Definition 1. The unperturbed motion ${}^t x_i(a_j)$ is stable with respect to parameters a_j if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|{}^{t_0} x_i(a_j) - {}^{t_0} \bar{x}_i(\bar{a}_j)| < \epsilon \tag{1}$$

implies

$$|{}^t x_i(a_j) - {}^t \bar{x}_i(\bar{a}_j)| < \delta. \tag{2}$$

Definition 2. The unperturbed motion ${}^t x_i$ is called quasi-asymptotically stable if the condition

$$\lim_{t \to \infty} \{{}^t x_i(a_j) - {}^t \bar{x}_i(\bar{a}_j)\} = 0 \tag{3}$$

holds for all perturbed values of $\bar{a}_j$ in the neighborhood of a_j.

Definition 3. The unperturbed motion is called asymptotically stable if it is both stable and quasi-asymptotically stable. A motion which is stable but not asymptotically stable is called weakly stable.

Let the motion of the system be described by a first order differential equation of the form

$$ {}^t \dot{x}_i = f_i({}^t x_j, t). \tag{4}$$

Also, let the perturbed motion be represented as

$$ {}^t \bar{x}_i = {}^t x_i + \bar{u}_i, \tag{5}$$

where $\bar{u}_i$ is the offset motion from the unperturbed state. The perturbed motions must also satisfy eqn (4) provided the motion ${}^t x_i$ is replaced by ${}^t \bar{x}_i$. After substituting eqn (5) into eqn (4), the following equation is obtained:

$$\dot{\bar{u}}_i = f_i({}^t x_j + \bar{u}_j, t) - f_i({}^t x_j, t) = g_i(\bar{u}_j, t), \tag{6}$$

with the condition $g_i(0, t) = 0$. Equation (6) is called the differential equation of the perturbed motion with $\bar{u}_i = 0$ as its trivial solution. If the functions g_i are only implicitly dependent on time t, i.e. $g_i = g_i(\bar{u}_j)$, the system is called autonomous, otherwise it is called nonautonomous. If eqn (4) gives a constant solution ${}^t x_i = c_i$ for all $t \geq t_0$, the system is said to be in a state of equilibrium, i.e. ${}^t \dot{x}_i = 0$.

Definition 4. A function $f(x, t)$ is called decrescent, if it satisfies $|f(x, t)| \leq \psi(|x|)$, where $\psi(r)$ is continuous real function in a close interval which vanishes at $r = 0$, i.e. $\psi(0) = 0$, and increases strictly monotonically with r.

2.2. *Liapunov's direct (second) method*

Liapunov's stability theorem. The solution of the differential eqn (4) is stable if there exists a positive definite (or negative definite) function $A(\bar{u}_i, t)$ such that its total time derivative $\dot{A}$ with $\dot{\bar{u}}_i$ given in eqn (6) is nonpositive (or, non-negative), i.e.

$$\dot{A} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial \bar{u}_i}\dot{\bar{u}}_i = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial \bar{u}_i} g_i(\bar{u}_j, t) \leq 0 \ \text{(or } \geq 0). \tag{7}$$

In addition, if $A(\bar{u}_i, t)$ is also a decrescent function, the solution is asymptotically stable [2, 4, 5].

Note that a summation on the range of the repeated index is implied in eqn (7). This convention is used throughout the paper.

Liapunov's instability theorem. The solution is unstable if there exists a positive definite decrescent (or negative definite decrescent) function $A(\bar{u}_i, t)$ whose total derivative with $\dot{\bar{u}}_i$ given in eqn (6) is positive (or negative) [2, 4, 5].

$$\dot{A} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial \bar{u}_i}\dot{\bar{u}}_i = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial \bar{u}_i} g_i(\bar{u}_j, t) > 0 \ \text{(or } < 0). \tag{8}$$

The functions $A(\bar{u}_i, t)$ that satisfy either of the foregoing theorem are called Liapunov functions. It should be noted that, for each problem, the choice of Liapunov function is not unique.

2.3. *Stability of autonomous systems*

For autonomous systems, the differential eqns (4) and (6) reduce to the form

$$ {}^t \dot{x}_i = f_i({}^t x_j) \tag{9}$$

$$\dot{\bar{u}}_i = f_i({}^t x_j + \bar{u}_j) - f_i({}^t x_j) = g_i(\bar{u}_j). \tag{10}$$

In general, eqns (9) and (10) are nonlinear. However, if the unperturbed state is stable, the perturbed motion stays in a neighborhood of the unperturbed state. Thus, eqn (10) can be approximated using Taylor's expansion in the neighborhood of $\bar{u}_i = 0$ as

$$\dot{\bar{u}}_i = \frac{\partial f_i({}^t x_k)}{\partial {}^t x_j} \bar{u}_j + \text{h.o.t.} = a_{ij} \bar{u}_j, \tag{11}$$

where $a_{ij} = \partial f_i({}^t x_k)/\partial {}^t x_j$. The matrix $[a_{ij}]$ is not necessarily symmetric. With the assumption of stable motion, there exists a function $A(\bar{u}_i)$ which satisfies the Liapunov stability theorem. Without loss of generality, let such a function be the positive definite quadratic form

$$A(\bar{u}_i) = b_{ij} \bar{u}_i \bar{u}_j, \tag{12}$$

where the matrix $[b_{ij}]$ is symmetric and positive definite. The stability condition implies that total time derivatives of A along every trajectory of eqn (11) must be negative, i.e.

$$\dot{A} = 2b_{ij} \bar{u}_i \dot{\bar{u}}_j = 2b_{ij} a_{jk} \bar{u}_i \bar{u}_k < 0. \tag{13}$$

That is, the matrix $[b_{ij} a_{jk}]$ must be negative semi-definite or definite. Since the matrix $[b_{ij}]$ is positive definite, this condition can be satisfied if and only if all the eigenvalues of the matrix $[a_{ij}]$ have a non-positive real part [5, 45, 46]. Let ω_i, $i = 1 \ldots n$, be the eigenvalues of matrix $[a_{ij}]$. Then, one has

$$\mathrm{Re}(\omega_i) \leqslant 0; \quad i = 1 \ldots n \tag{14}$$

as the stability condition. The system of eqn (9) becomes unstable when real part of one of the eigenvalues of $[a_{ij}]$ becomes positive. Therefore, the critical state, which is a transition between stable and unstable states, is governed by

$$\mathrm{Re}(\omega_i) = 0; \quad \text{for any } i, \quad i = 1 \ldots n. \tag{15}$$

2.4. *Stability of nonautonomous systems*

The stability investigation of nonautonomous systems described by eqn (4) with equations of the perturbed motion in eqn (6), in general, requires a more profound approach. However, simple conclusions can be drawn for some systems of equations using a method parallel to the one in the previous section. If the matrix $[a_{ij}]$, which is now function of t, i.e. $a_{ij} = a_{ij}(t)$, is bounded and if the eigenvalues of $[a_{ij}]$ all have nonpositive real parts for every fixed $t \geqslant t_0$, sufficient stability can be obtained [5].

2.5. *Application of Liapunov's direct (second) method to conservative systems*

The conservative system imples that

$$-\frac{\partial {}^t H}{\partial {}^t x_i} = {}^t\dot{p}_i; \quad \frac{\partial {}^t H}{\partial {}^t p_i} = {}^t\dot{x}_i,$$

where ${}^tH$ is the total energy of the mechanical system at time t, i.e. ${}^tH = {}^tT + {}^t\pi$, where ${}^tT = {}^tT({}^t\dot{x}_i)$ is the kinematic energy, ${}^t\pi = {}^t\pi({}^tx_i)$ is the potential energy, and ${}^tp_i$ is the momentum. Thus, the perturbed motion for this system can be written in a neighborhood of the unperturbed state as

$$-\frac{\partial {}^t\bar{H}}{\partial {}^t\bar{x}_i} = {}^t\dot{\bar{p}}_i, \tag{16}$$

where

$${}^t\bar{x}_i = {}^tx_i + \bar{u}_i$$

$${}^t\bar{p}_i = {}^tp_i + \bar{p}_i$$

$${}^t\bar{H}({}^t\bar{x}_i, {}^t\bar{p}_i) = {}^tH({}^tx_i, {}^tp_i) + \bar{H}.$$

Writing Taylor's expansion about $\bar{u}_i = 0$ for the left hand side of eqn (16) and neglecting higher order terms, the equation for the perturbed motion is approximated as

$$-\frac{\partial^2({}^tH)}{\partial {}^tx_i \partial {}^tx_j} \bar{u}_j = \dot{\bar{p}}_i. \tag{17}$$

From Sec. 2.3, the motion of system is stable if and only if the matrix

$$\left[\frac{\partial^2({}^tH)}{\partial {}^tx_i \partial {}^tx_j}\right]$$

has all eigenvalues with positive real parts. Since

$$\left[\frac{\partial^2({}^tH)}{\partial {}^tx_i \partial {}^tx_j}\right]$$

is symmetric, all the eigenvalues are real. And since

$$\left[\frac{\partial^2({}^tH)}{\partial {}^tx_i \partial {}^tx_j}\right] = \left[\frac{\partial^2({}^t\pi)}{\partial {}^tx_i \partial {}^tx_j}\right],$$

one can conclude that the motion of a conservative mechanical system is stable if and only if the potential energy has an isolated local minimum. Moreover, the motion ceases to be stable when the Hessian of potential energy becomes indefinite. Thus, the stability criterion for such systems is

$$\det\left[\frac{\partial^2({}^{t^*}\pi)}{\partial {}^tx_i \partial {}^tx_j}\right] = 0, \tag{18}$$

where t^* denotes time or load level for the critical state.

3. STABILITY OF STRUCTURAL SYSTEMS

3.1. *Stability criterion for conservative systems*

It has been shown in Sec. 2.5 that the condition for stability of conservative systems is the existence of a local minimum for the potential energy. In the variational form, this is equivalent to

$$\delta^2({}^t\pi) > 0. \tag{19}$$

The system becomes unstable whenever $\delta^2({}^t\pi)$ ceases to be positive definite. The stability limit, denoted by the state t^*, is then governed by the following condition:

$$\delta^2({}^{t^*}\pi) = 0. \tag{20}$$

Using the undeformed configuration for reference, the potential energy of structural systems can be expressed as

$${}^t\pi = \int_{{}^0v} U_0({}^t_0\epsilon_{ij})\,{}^0\mathrm{d}V - \int_{{}^0v} {}^0\rho\,{}^t_0f_i\,{}^tu_i\,{}^0\mathrm{d}V - \int_{{}^0\Gamma_T} {}^t_0T_i\,{}^tu_i\,{}^0\mathrm{d}\Gamma_T, \tag{21}$$

where $U_0({}^t_0\epsilon_{ij})$ denotes the internal energy per unit volume, called strain energy density, ${}^tu_i$ is the displacement field, ${}^t_0\epsilon_{ij}$ are the Cartesian components of the Green–Lagrange strain tensor at time t referred to the configuration at time zero, i and j have values 1, 2 and 3, and the left superscript refers to the configuration in which the quantity occurs, and the left subscript refers to a reference configuration for the quantity [47]. Later, the left superscript on some quantities will be omitted to indicate that they are only increments. Note that existence of a strain energy density function is assumed, so the formulation applies to linearly elastic, nonlinear elastic and hyperelastic materials. These are important materials in many practical applications. Note also that full kinematic nonlinearities are included in the formulation. The first variation of eqn (21) is given as

$$\delta{}^t\pi = \int_{{}^0v} ({}^t_0S_{ij}\delta{}^t_0\epsilon_{ij} - {}^0\rho\,{}^t_0f_i\delta{}^tu_i)\,{}^0\mathrm{d}V - \int_{{}^0\Gamma_T} {}^t_0T_i\delta{}^tu_i\,{}^0\mathrm{d}\Gamma_T, \tag{22}$$

where ${}^t_0S_{ij}$ are the Cartesian components of the second Piola–Kirchhoff stress tensor corresponding to the configuration at time t but referred to the initial configuration. The Green–Lagrange strain tensor ${}^t_0\epsilon_{ij}$ used with the second Piola–Kirchhoff stress tensor is defined as:

$${}^t_0\epsilon_{ij} = \frac{1}{2}\left\{\frac{\partial{}^tu_i}{\partial{}^0x_j} + \frac{\partial{}^tu_j}{\partial{}^0x_i} + \frac{\partial{}^tu_k}{\partial{}^0x_i}\frac{\partial{}^tu_k}{\partial{}^0x_j}\right\}, \tag{23}$$

and its arbitrary variation is given as

$$\delta{}^t_0\epsilon_{ij} = {}_0\mathrm{e}_{ij}({}^t\mathbf{u}, \delta{}^t\mathbf{u})$$

$${}_0\mathrm{e}_{ij}({}^t\mathbf{u}, \mathbf{a}) = \frac{1}{2}\left\{\frac{\partial a_i}{\partial{}^0x_j} + \frac{\partial a_j}{\partial{}^0x_i} + \frac{\partial{}^tu_k}{\partial{}^0x_i}\frac{\partial a_k}{\partial{}^0x_j} + \frac{\partial a_k}{\partial{}^0x_i}\frac{\partial{}^tu_k}{\partial{}^0x_j}\right\}. \tag{24}$$

Note that ${}_0\mathrm{e}_{ij}({}^t\mathbf{u}, \mathbf{a})$ are linear operators defined on the field $\mathbf{a}$. The second Piola–Kirchhoff stresses are related to Cauchy stresses as

$${}^t_0S_{ij} = \frac{{}^0\rho}{{}^t\rho}\,{}^0_tx_{i,s}\,{}^t\tau_{rs}\,{}^0_tx_{j,r},$$

where ${}^0_tx_{i,s}$ is the inverse of the deformation gradient tensor, $({}^0_tx_{i,s}) = ({}^t_0x_{i,s})^{-1}$, and ${}^t\tau_{rs}$ are Cauchy stresses.

Using eqn (22), the second variation of the total potential energy is given as

$$\delta^2({}^t\pi) = \int_{{}^0v} \{\delta{}^t_0S_{ij}\delta{}^t_0\epsilon_{ij} + {}^t_0S_{ij}\delta(\delta{}^t_0\epsilon_{ij}) - {}^0\rho\,{}^t_0f_i\delta^2({}^tu_i)\}\,{}^0\mathrm{d}V - \int_{{}^0\Gamma_T} {}^t_0T_i\delta^2({}^tu_i)\,{}^0\mathrm{d}\Gamma_T. \tag{25}$$

To account for nonlinear material behavior, the following rate or incremental form of the constitutive law is used:

$${}^t_0\dot{S}_{ij} = {}^t\phi_{ijkl}\,{}^t_0\dot{\epsilon}_{kl}, \tag{26}$$

where ${}^t\phi_{ijkl}$ is the tangent modulus tensor that may depend on total stress, total strain, and strain history. Also, from eqn (24), $\delta(\delta{}^t_0\epsilon_{ij})$ can be obtained as

$$\delta(\delta{}^t_0\epsilon_{ij}) = {}_0\mathrm{e}_{ij}({}^t\mathbf{u}, \delta^2({}^t\mathbf{u})) + {}_0\eta_{ij}(\delta{}^tu_k, \delta{}^tu_k), \tag{27}$$

where ${}_0\eta_{ij}(a_k, b_k)$ are the nonlinear operators defined on a_k and b_k as

$${}_0\eta_{ij}(a_k, b_k) = \frac{1}{2}\left\{\frac{\partial a_k}{\partial{}^0x_i}\frac{\partial b_k}{\partial{}^0x_j} + \frac{\partial b_k}{\partial{}^0x_i}\frac{\partial a_k}{\partial{}^0x_j}\right\}. \tag{28}$$

Substituting eqns (26) and (27) into eqn (25), $\delta^2({}^t\pi)$ becomes

$$\delta^2({}^t\pi) = \left[\int_{{}^0v} {}^t\phi_{ijkl}\delta{}^t_0\epsilon_{kl}\delta{}^t_0\epsilon_{ij} + {}^t_0S_{ij}\,{}_0\eta_{ij}(\delta{}^tu_k, \delta{}^tu_k)\,{}^0\mathrm{d}V\right] + \left[\int_{{}^0v} {}^t_0S_{ij}\,{}_0\mathrm{e}_{ij}({}^t\mathbf{u}, \delta^2({}^t\mathbf{u})) - {}^0\rho\,{}^t_0f_i\delta^2({}^tu_i)\,{}^0\mathrm{d}V - \int_{{}^0\Gamma_T} {}^t_0T_i\delta^2({}^tu_i)\,{}^0\mathrm{d}\Gamma_T\right]. \tag{29}$$

Note that the terms in the last bracket in eqn (29) are similar to the equilibrium equation ($\delta{}^t\pi = 0$) if a special variation $\delta^2({}^tu_i) = \delta{}^tu_i$ is selected. Thus, combining eqns (20) and (29), one obtains the stability criterion for conservative systems, which is known as the Euler method of investigation stability, as

$$\int_{{}^0v} \{{}^{t^*}\phi_{ijkl}\delta{}^{t^*}_0\epsilon_{kl}\delta{}^{t^*}_0\epsilon_{ij} + {}^{t^*}_0S_{ij}\,{}_0\eta_{ij}(\delta{}^{t^*}u_k, \delta{}^{t^*}u_k)\}\,{}^0\mathrm{d}V = 0. \tag{30}$$

where t^* represents the critical state.

It should be pointed out that the essential part of Euler's method is the existence of a potential function of the external forces. The method is applicable if the external forces have a potential and, in general, it is not if they do not. A discretized form of the method has been used along with the incremental formulation to determine the critical load [44] and in imposing the stability constraint on structural systems in their optimum design.

3.2. *Stability criterion for nonconservative systems*

Since nonconservative forces do not possess potential functional, the principle of minimum potential energy fails to establish the point of transition from the stable to unstable state [32, 39]. However, the concept of adjacent states is not bounded by these restrictions [39]. The guiding philosophy is to extend the familiar variational principle of dynamics, the Hamilton's principle, to the nearby perturbed state of motion. In this extension the results already known for the unperturbed state are used to drop some terms. Also, Taylor's expansion is used to refer all the quantities appearing in the adjacent state of motion to the corresponding known quantities at the unperturbed state. By investigating the subsequent motions of the adjacent state around the unperturbed state a general criterion for stability is obtained.

Sections 3.2.1 and 3.2.2 set up the equations of motion for the unperturbed state and adjacent states. In Sec. 3.2.3, the subsequent motions of the adjacent state are studied and the stability criterion is formulated. In Sec. 3.2.4, the analysis leading to the distinction between the bifurcation point and limit point is presented.

3.2.1. *Equations of motion for the structural system.* For any mechanical system in equilibrium or motion, Hamilton's principle states that for any kinematically admissible deviation from a fundamental state, one has

$$\delta A(t)=\int_{t_0}^{t_1}(\delta' T-\delta'\pi+\delta' W_{nc})\,\mathrm{d}t=0, \quad (31)$$

where t_0 and t_1 are the initial and final times; and $\delta' W_{nc}$ is the virtual work of nonconservative forces. Here $\delta A(t)$ is not the first variation of any action functional as in variational calculus; it is simply a collection of the first-order terms that result by applying the d'Alembert principle to the state t. In the continuum form, $\delta' T$, $\delta'\pi$, and $\delta' W_{nc}$ are given as

$$\delta' T=\int_{{}^0V}{}^0\rho\,{}^t\ddot{u}_i\,\delta{}^t\dot{u}_i\,{}^0\mathrm{d}V$$

$$\delta'\pi=\int_{{}^0V}({}^t_0S_{ij}\,\delta{}^t_0\epsilon_{ij}-{}^0\rho\,{}^t_0f_i\,\delta{}^tu_i)\,{}^0\mathrm{d}V-\int_{{}^0\Gamma_T}{}^t_0T_i\,\delta{}^tu_i\,{}^0\mathrm{d}\Gamma_T$$

$$\delta' W_{nc}=\int_{{}^0V}{}^0\rho\,{}^tq_i\,\delta{}^tu_i\,{}^0\mathrm{d}V+\int_{{}^0\Gamma_T}{}^t_0x_{i,j}\,{}^t_0Q_j\,\delta{}^tu_i\,{}^0\mathrm{d}\Gamma_T, \quad (32)$$

where ${}^0\rho$ is the mass density at the undeformed state, ${}^t_0f_i$ and ${}^tq_i$ are components of the conservative and nonconservative body forces per unit mass at state t measured with respect to the undeformed configuration, and ${}^t_0T_i$ and ${}^t_0Q_i$ are the components of the conservative and nonconservative applied surface forces at state t measured with respect to the undeformed configuration, respectively. Combining eqns (30) and (31) and integrating by parts, one obtains

$$\delta A(t)=\int_{t_0}^{t_1}\int_{{}^0V}(-{}^0\rho\,{}^t\ddot{u}_i\,\delta{}^tu_i-{}^t_0S_{ij}\,\delta{}^t_0\epsilon_{ij}+{}^0\rho\,{}^t_0f_i\,\delta{}^tu_i+{}^0\rho\,{}^tq_i\,\delta{}^tu_i)\,{}^0\mathrm{d}V\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{{}^0\Gamma_T}({}^t_0T_i\,\delta{}^tu_i+{}^t_0x_{i,j}\,{}^t_0Q_j\,\delta{}^tu_i)\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t=0. \quad (33)$$

The above equation represents the equation of motion in Total Lagrangian formulation for any mechanical system.

3.2.2. *Equation of motion for the adjacent state.* Similar to the procedure described in Sec. 3.2.1, the adjacent state of motion at t must be governed by eqn (33) provided the perturbed quantities (denoted by an overbar) are used, i.e.

$$\delta\bar{A}(t)=\int_{t_0}^{t_1}\int_{{}^0V}(-{}^0\rho\,{}^t\ddot{\bar{u}}_i\,\delta{}^t\bar{u}_i-{}^t_0\bar{S}_{ij}\,\delta{}^t_0\bar{\epsilon}_{ij}+{}^0\rho\,{}^t\bar{f}_i\,\delta{}^t\bar{u}_i+{}^0\rho\,{}^t\bar{q}_i\,\delta{}^t\bar{u}_i)\,{}^0\mathrm{d}V\,\mathrm{d}t+\int_{t_0}^{t_1}\int_{{}^0\Gamma_T}({}^t_0\bar{T}_i\,\delta{}^t\bar{u}_i+{}^t_0\bar{x}_{i,j}\,{}^t_0\bar{Q}_j\,\delta{}^t\bar{u}_i)\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t=0. \quad (34)$$

Since the adjacent state is unknown, the following decompositions are needed:

$${}^t\bar{u}_i={}^tu_i+\bar{u}_i$$

$${}^t_0\bar{S}_{ij}={}^t_0S_{ij}+{}_0\bar{S}_{ij}$$

$${}^t_0\bar{\epsilon}_{ij}={}^t_0\epsilon_{ij}+{}_0\bar{\epsilon}_{ij}, \quad (35)$$

where ${}_0\bar{\epsilon}_{ij}={}_0e_{ij}({}^t\mathbf{u},\bar{\mathbf{u}})+\frac{1}{2}{}_0\eta_{ij}(\bar{u}_k,\bar{u}_k)$. The quantities without left superscript t are increments from configuration at time t to the perturbed configuration without increase in the load intensity. ${}_0e_{ij}({}^t\mathbf{u},\bar{\mathbf{u}})$ and $\frac{1}{2}{}_0\eta_{ij}(\bar{u}_k,\bar{u}_k)$ are the linear and geometrically nonlinear strain increments with respect to the increment in the displacements, respectively. These operators are defined in eqns (24) and (28), respectively. The variations of displacements and Green–Lagrange strain tensor in the adjacent state are given as

$$\delta{}^t\bar{u}_i=\delta\bar{u}_i$$

$$\delta{}^t_0\bar{\epsilon}_{ij}=\delta_0\bar{\epsilon}_{ij}={}_0e_{ij}({}^t\mathbf{u},\delta\bar{\mathbf{u}})+{}_0\eta_{ij}(\bar{u}_k,\bar{u}_k). \quad (36)$$

Since the perturbed and unperturbed states are under the same loading intensity, ${}^t_0\bar{f}_i$ and ${}^t_0\bar{T}_i$ are equal to ${}^t_0f_i$ and ${}^t_0T_i$, respectively. However, the nonconservative forces at the two states are not the same because the loads are configuration dependent. Therefore, nonconservative forces are approximated by linear Taylor's expansion as

$$
{}^t\bar{q}_i \approx {}^tq_i + \frac{\partial {}^tq_i}{\partial {}^tu_j}\bar{u}_j + \frac{\partial {}^tq_i}{\partial {}^t\dot{u}_j}\dot{\bar{u}}_j
$$

$$
{}^t_0\bar{Q}_i \approx {}^t_0Q_i + \frac{\partial {}^t_0Q_i}{\partial {}^tu_j}\bar{u}_j + \frac{\partial {}^t_0Q_i}{\partial {}^t\dot{u}_j}\dot{\bar{u}}_j. \tag{37}
$$

Substituting eqns (35)–(37) into eqn (34) and noting that ${}^t\bar{x}_i = {}^tx_i + \bar{u}_i$, the following equation is obtained:

$$
\begin{aligned}
\delta\bar{A}(t) = {} & \Bigg[\int_{t_0}^{t_1}\int_{{}^0V}\{-{}^0\rho\,{}^t\ddot{u}_i\delta\bar{u}_i - {}^t_0S_{ij}\,{}_0e_{ij}({}^t\mathbf{u},\delta\bar{\mathbf{u}}) + {}^0\rho\,{}^t_0f_i\delta\bar{u}_i \\
& + {}^0\rho\,{}^tq_i\delta\bar{u}_i\}\,{}^0\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\{{}^t_0T_i\delta\bar{u}_i \\
& + {}^t_0x_{l,j}\,{}^t_0Q_j\delta\bar{u}_i\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] \\
& + \Bigg[\int_{t_0}^{t_1}\int_{{}^0V}\Big\{-{}^0\rho\ddot{\bar{u}}_i\delta\bar{u}_i \\
& - ({}^t_0S_{ij}\,{}_0\eta_{ij}(\bar{u}_k,\delta\bar{u}_k) + {}_0\bar{S}_{ij}\,{}_0e_{ij}({}^t\mathbf{u},\delta\bar{\mathbf{u}})) \\
& + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^tu_j}\bar{u}_j\delta\bar{u}_i + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^t\dot{u}_j}\dot{\bar{u}}_j\delta\bar{u}_i\Big\}\,{}^0\mathrm{d}V\,\mathrm{d}t \\
& + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial\bar{u}_i}{\partial {}^0x_j}\,{}^t_0Q_j\delta\bar{u}_i + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^tu_k}\bar{u}_k\delta\bar{u}_i \\
& + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^t\dot{u}_k}\dot{\bar{u}}_k\delta\bar{u}_i\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] \\
& + \Bigg[\int_{t_0}^{t_1}\int_{{}^0V}{}_0\bar{S}_{ij}\,{}_0\eta_{ij}(\bar{u}_k,\delta\bar{u}_k)\,{}^0\mathrm{d}V\,\mathrm{d}t \\
& + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial\bar{u}_i}{\partial {}^0x_j}\frac{\partial {}^t_0Q_j}{\partial {}^tu_k}\bar{u}_k\delta\bar{u}_i \\
& + \frac{\partial\bar{u}_i}{\partial {}^0x_j}\frac{\partial {}^t_0Q_j}{\partial {}^t\dot{u}_k}\dot{\bar{u}}_k\delta\bar{u}_i\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] = 0.
\end{aligned} \tag{38}
$$

Note that $\bar{u}_i$ (and $\delta\bar{u}_i$) must satisfy the prescribed displacements, i.e. $\bar{u}_i$ (and $\delta\bar{u}_i$) is a kinematically admissible field. Thus, the first bracket in eqn (38) vanishes since it satisfies the Hamilton principle at the state t. Also, terms in the last bracket are neglected because they are of higher order than other terms. Therefore, eqn (38) reduces to

$$
\begin{aligned}
\delta\bar{A}(t) = {} & \int_{t_0}^{t_1}\int_{{}^0V}\Big\{-{}^0\rho\ddot{\bar{u}}_i\delta\bar{u}_i - ({}^t_0S_{ij}\,{}_0\eta_{ij}(\bar{u}_k,\delta\bar{u}_k) \\
& + {}_0\bar{S}_{ij}\,{}_0e_{ij}({}^t\mathbf{u},\delta\bar{\mathbf{u}}) + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^tu_j}\bar{u}_j\delta\bar{u}_i \\
& + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^t\dot{u}_j}\dot{\bar{u}}_j\delta\bar{u}_i\Big\}\,{}^0\mathrm{d}V\,\mathrm{d}t \\
& + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial\bar{u}_i}{\partial {}^0x_j}\,{}^t_0Q_j\delta\bar{u}_i + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^tu_k}\bar{u}_k\delta\bar{u}_i \\
& + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^t\dot{u}_k}\dot{\bar{u}}_k\delta\bar{u}_i\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t = 0,
\end{aligned} \tag{39}
$$

which represents the equation of motion for the adjacent state.

3.2.3. *General criterion for structural stability.* To establish the general nonlinear stability criterion, the subsequent motion of the adjacent state must be observed by introducing

$$
\bar{u}_i = \tilde{u}_i e^{\omega t};\ \dot{\bar{u}}_i = \omega\tilde{u}_i e^{\omega t};\ \ddot{\bar{u}}_i = \omega^2\tilde{u}_i e^{\omega t};\ \delta\bar{u}_i = \delta\tilde{u}_i e^{\omega t}, \tag{40}
$$

where $\tilde{u}_i$ are amplitudes of $\bar{u}_i$. Also, from the general stress–strain constitutive law in eqn (26), the incremental form after linearization may be written as

$$
{}_0\bar{S}_{ij} = {}^t\phi_{ijkl}\,{}_0e_{kl}({}^t\mathbf{u},\bar{\mathbf{u}}). \tag{41}
$$

It is important to note that the incremental stress–strain law in eqn (26) can be used to represent phenomena like plasticity, creep, viscoelasticity, and viscoplasticity. These are classified as internally or materially nonconservative problems.

Substituting eqns (40) and (41) into eqn (39) and noting that $e^{2\omega t}$ is arbitrary yields the stability criterion for the general structural system as

$$
\begin{aligned}
& \int_{{}^0V}\Big\{-{}^0\rho\tilde{u}_i\delta\tilde{u}_i\omega^2 - ({}^t_0S_{ij}\,{}_0\eta_{ij}(\tilde{u}_k,\delta\tilde{u}_k) \\
& + {}^t\phi_{ijkl}\,{}_0e_{kl}({}^t\mathbf{u},\tilde{\mathbf{u}})\,{}_0e_{ij}({}^t\mathbf{u},\delta\tilde{\mathbf{u}}) \\
& + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^tu_j}\tilde{u}_j\delta\tilde{u}_i + {}^0\rho\frac{\partial {}^tq_i}{\partial {}^t\dot{u}_j}\tilde{u}_j\delta\tilde{u}_i\omega\Big\}\,{}^0\mathrm{d}V \\
& + \int_{{}^0\Gamma_T}\Big\{\frac{\partial\tilde{u}_i}{\partial {}^0x_j}\,{}^t_0Q_j\delta\tilde{u}_i + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^tu_k}\tilde{u}_k\delta\tilde{u}_i \\
& + {}^t_0x_{l,j}\frac{\partial {}^t_0Q_j}{\partial {}^t\dot{u}_k}\tilde{u}_k\delta\tilde{u}_i\omega\Big\}\,{}^0\mathrm{d}\Gamma_T = 0.
\end{aligned} \tag{42}
$$

Equation (42) gives the load–frequency nonself-adjoint quadratic eigenvalue problem. Because of the asymmetry, the ωs are, in general, complex, i.e.

$$
\omega = \alpha + i\beta, \tag{43}
$$

where α and β are real and imaginary parts of ω. Thus, as time increases indefinitely, $\bar{u}_i$ reduce in magnitude and the adjacent state moves toward the state of motion at t, if and only if all the αs are negative. Moreover, if α is zero, the adjacent state keeps oscillating around the state at t. Therefore, the condition for stability of nonconservative systems is

$$\alpha = \mathrm{Re}(\omega) \leqslant 0 \tag{44}$$

for all ω satisfying eqn (42). If any α is positive, we see from eqn (40) that $\bar{u}_i$ increases as time increases and the adjacent state moves away from the state at t. For this case, the system is in an unstable condition. The stable zone for ωs can be clearly shown in a complex plane as shown in Fig. 1. Furthermore, the transition between stable and unstable states is along the β axis. Thus, at the critical point, the following condition must hold:

$$\alpha = 0. \tag{45}$$

However, condition (45) does not give the critical load. The critical load can be found by increasing load intensity in eqn (42) and monitoring the point where ω ceases to satisfy the condition (44). If the critical configuration is denoted by t^* and the stability criterion in eqn (42) becomes

$$\int_{{}^0V}\left\{-{}^0\rho\bar{u}_i\delta\bar{u}_i\omega^2-({}^{t^*}_{0}S_{ij}\,{}_0\eta_{ij}(\bar{u}_k,\delta\bar{u}_k)\right.$$
$$+{}^{t^*}\phi_{ijkl}\,{}_0e_{kl}({}^{t^*}\mathbf{u},\bar{\mathbf{u}})_0e_{ij}({}^{t^*}\mathbf{u},\delta\bar{\mathbf{u}})$$
$$\left.+{}^0\rho\frac{\partial\,{}^{t^*}q_i}{\partial\,{}^tu_j}\bar{u}_j\delta\bar{u}_i+{}^0\rho\frac{\partial\,{}^{t^*}q_i}{\partial\,{}^t\dot{u}_j}\bar{u}_j\delta\bar{u}_i\omega\right\}{}^0\mathrm{d}V$$
$$+\int_{{}^0\Gamma_T}\left\{\frac{\partial\bar{u}_i}{\partial\,{}^0x_j}\,{}^{t^*}_{0}Q_j\delta\bar{u}_i+{}^{t^*}_{0}x_{i,j}\frac{\partial\,{}^{t^*}_{0}Q_j}{\partial\,{}^tu_k}\bar{u}_k\delta\bar{u}_i\right.$$
$$\left.+{}^{t^*}_{0}x_{i,j}\frac{\partial\,{}^{t^*}_{0}Q_j}{\partial\,{}^t\dot{u}_k}\bar{u}_k\delta\bar{u}_i\omega\right\}{}^0\mathrm{d}\Gamma_T=0. \tag{46}$$

For a problem without damping, the terms associated with the derivative with respect to velocity vanish. Therefore, the eigenvalue problem of eqn (46) gives values for ω^2, instead of ω, which can be either real or complex. The conditions for ω^2 can be divided into three cases; complex number, positive real number, and negative real number. If one of the ω^2s is a complex number, the structure is unstable. This is because the square root of complex numbers always yields two complex values, one of which has a positive real part. Secondly, if one of the ω^2s is real and positive, then, the structure is again unstable. Only the situation where all ω^2s are negative real numbers implies stability of the system. Hence the negative real value of ω^2 provides the necessary and sufficient condition for stability of undamped systems. The critical point occurs when an ω^2 ceases to

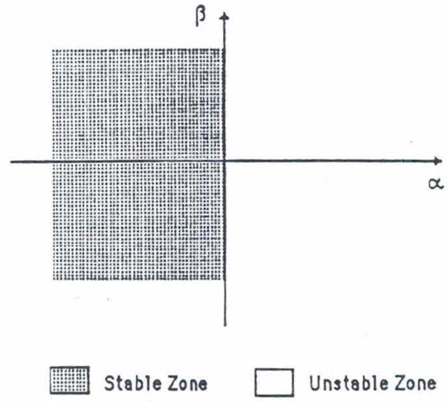

Fig. 1. Stable zone in complex plane for ω.

be a negative real number. Note that for nonconservative systems, the transition does not necessarily occur when $\omega^2 = 0$, since ω^2s may become complex before any of them reach zero value. The stable zone in the ω^2s complex plane is shown in Fig. 2.

It is important to note that the eigenvalue problem [eqn (46)] involves two parameters, ω and the loads. Therefore, an iterative procedure must be used to determine the critical loads. The procedure would involve repeatedly changing the loads and solving the eigenvalue problem for ω or ω^2, as the case may be, until a critical point condition, as discussed in the foregoing, is satisfied. This procedure will be demonstrated in a later paper using a numerical example.

If the systems are internally nonconservative only, i.e. there are no nonconservative external forces and if the material is isotropic, the eigenvalue problem (46) becomes symmetric. This is due to the fact that the material modulus is symmetric. Thus, the eigenvalues of eqn (46) are all real and the one that satisfies eqn (45) must be zero (i.e. $\omega^2 = 0$). Furthermore, since $\bar{u}_i$ and $\delta\bar{u}_i$ vanish along the displacement prescribed boundary—namely they are kinematically admissible—$\bar{u}_i$ and $\delta\bar{u}_i$ are interchangeable with $\delta\,{}^{t^*}u_i$ (or δu_i). As a result, the stability criterion for systems with isotropic, nonconservative (or conservative) materials reduces to the criterion for conservative

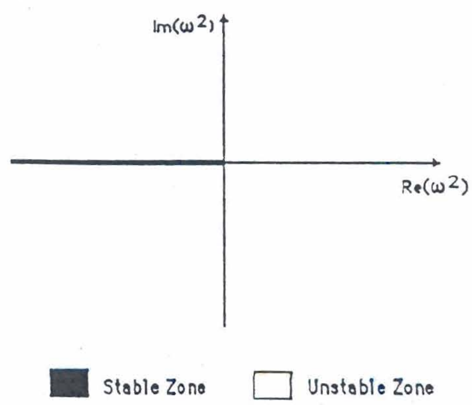

Fig. 2. Stable zone in complex plane for ω^2.

systems given in eqn (30). This explains that the Euler stability criterion is applicable to internally nonconservative systems with isotropic material.

3.2.4. *Limit point and bifurcation point analysis.* A criterion to distinguish between the limit point and bifurcation point instability was presented for finite dimensional conservative systems by Huseyin [30] and Riks [48]. The basic idea of an analysis to derive such a criterion is to determine whether or not another equilibrium state in a neighborhood of the critical state is possible at a load slightly over the critical load. If such a state is possible, then the critical load determined using the criterion in eqn (46) corresponds to bifurcation point instability; otherwise, it corresponds to the limit point instability. To use the foregoing procedure for a finite dimensional quasi-static system, let ${}^{t^*}\mathbf{F}$ and ${}^{t^*}\mathbf{R}$ be the internal and external forces at the critical point. Thus, ${}^{t^*}\mathbf{F} - {}^{t^*}\mathbf{R} = 0$. Now the load is given an increment $\Delta\mathbf{R}$ and an equilibrium state at ${}^{t^*}\mathbf{R} + \Delta\mathbf{R}$ is desired. The new state will be in equilibrium if the virtual work of all the forces in going through arbitrary, but kinematically admissible, virtual displacements is zero. In particular, we use the eigenvector representing the critical state as the virtual displacement vector in the virtual work principle. The internal forces in the virtual work equation are then approximated with respect to displacements using a linear Taylor's expansion about the critical state. This leads to the equation:

$$(y^T \hat{\mathbf{R}})\Delta\gamma = 0, \tag{47}$$

where y the eigenvector representing the critical state (obtained by solving the eigenvalue problem $\mathbf{K}_T y = 0$, where $\mathbf{K}_T$ is the tangent stiffness matrix at the critical point), $\hat{\mathbf{R}}$ is the normalized external load vector and $\Delta\gamma$ is the increment in the load parameter. In eqn (47), if $y^T\hat{\mathbf{R}} = 0$, then the critical point corresponds to bifurcation instability, because $\Delta\gamma$ can be positive indicating that the load level can be increased. On the other hand, if $y^T\hat{\mathbf{R}} \neq 0$, then $\Delta\gamma$ must be zero to satisfy eqn (47). This will indicate limit point instability of the system.

The foregoing procedure can be generalized to continua as well as nonconservative systems. This will give a variational form of the criterion to distinguish between bifurcation and limit point instability. The derivation procedure for the criterion will introduce an adjoint structure or variable, that has been widely used in the structural design sensitivity analysis [49–53]. The adjoint variables will replace the displacement variations around the critical state. Once the equation of motion has been expanded about the critical state, an adjoint equation will be identified, and criterion to distinguish between bifurcation and limit point instabilty will be obtained.

To simplify the problem, it is assumed that the forces acting on the structure are in a certain pattern. That is, there exists a load parameter, ${}^t\gamma$, such that

$$
{}^t_0f_i = {}^t_0f_i({}^t\gamma); \quad {}^t_0T_i = {}^t_0T_i({}^t\gamma)
$$

$$
{}^tq_i = {}^tq_i({}^tu_j, {}^t\dot{u}_k, {}^t\gamma); \quad {}^t_0Q_i = {}^t_0Q_i({}^tu_j, {}^t\dot{u}_k, {}^t\gamma). \tag{48}
$$

The load intensity depends on the load parameter in such a way that the load intensity is a monotonically increasing function of ${}^t\gamma$. A problem with unproportional loads can be viewed as a problem with several proportional loads.

Consider the system at load level slightly higher than the critical load, denoted by $t^* + \Delta t$. The motion of the system is governed by eqn (33), with the left superscript t replaced by $t^* + \Delta t$, i.e.

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{{}^0V} (-{}^0\rho\, {}^{t^*+\Delta t}\ddot{u}_i \delta\, {}^{t^*+\Delta t}u_i - {}^{t^*+\Delta t}_{0}S_{ij}\,\delta\, {}^{t^*+\Delta t}_{0}\epsilon_{ij} \\
&\quad + {}^0\rho\, {}^{t^*+\Delta t}_{0}f_i\,\delta\, {}^{t^*+\Delta t}u_i + {}^0\rho\, {}^{t^*+\Delta t}q_i\,\delta\, {}^{t^*+\Delta t}u_i)\, {}^0\mathrm{d}V\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T} ({}^{t^*+\Delta t}_{0}T_i\,\delta\, {}^{t^*+\Delta t}u_i \\
&\quad + {}^{t^*+\Delta t}_{0}x_{i,j}\, {}^{t^*+\Delta t}_{0}Q_j\,\delta\, {}^{t^*+\Delta t}u_i)\, {}^0\mathrm{d}\Gamma_T\,\mathrm{d}t = 0.
\end{aligned} \tag{49}
$$

Since $\delta\, {}^{t^*+\Delta t}u_i$ is any kinematically admissible displacement field, it is replaced by the displacement field u_i^a for an adjoint structure that has the same boundary conditions as the original structure. The strain field for the adjoint structure ${}^{t^*+\Delta t}\epsilon_{ij}^a$, that is compatible with the adjoint displacement field, is defined using variation of the Green–Lagrange strain tensor given in eqn (24) as [52, 53]:

$$
{}^{t^*+\Delta t}_{0}\epsilon_{ij}^a = {}_0\mathrm{e}_{ij}({}^{t^*+\Delta t}\mathbf{u}, \mathbf{u}^a).
$$

The equation of motion for the adjoint structure will be defined later. Thus, eqn (49) can be written as:

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{{}^0V} \{-{}^0\rho\, {}^{t^*+\Delta t}\ddot{u}_i u_i^a - {}^{t^*+\Delta t}_{0}S_{ij}\, {}^{t^*+\Delta t}_{0}\epsilon_{ij}^a \\
&\quad + {}^0\rho\, {}^{t^*+\Delta t}_{0}f_i u_i^a + {}^0\rho\, {}^{t^*+\Delta t}q_i u_i^a\}\, {}^0\mathrm{d}V\,\mathrm{d}t \\
&\quad + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T} \{{}^{t^*+\Delta t}_{0}T_i u_i^a \\
&\quad + {}^{t^*+\Delta t}_{0}x_{i,j}\, {}^{t^*+\Delta t}_{0}Q_j u_i^a\}\, {}^0\mathrm{d}\Gamma_T\,\mathrm{d}t = 0.
\end{aligned} \tag{50}
$$

Equation (50) can be further reduced by introducing the incremental decomposition as

$$
\begin{aligned}
{}^{t^*+\Delta t}u_i &= {}^{t^*}u_i + u_i \\
{}^{t^*+\Delta t}_{0}S_{ij} &= {}^{t^*}_0S_{ij} + {}_0S_{ij} \\
{}^{t^*+\Delta t}_{0}\epsilon_{ij}^a &= {}^{t^*}_0\epsilon_{ij}^a + {}_0\eta_{ij}(u_k, u_k^a).
\end{aligned} \tag{51}
$$

The applied load can also be decomposed in the neighborhood of critical point using Taylor's expansion for the load parameter, ${}^{t}\gamma$, as

$$
\begin{aligned}
{}^{t^*+\Delta t}_{0}f_i &\approx {}^{t^*}_{0}f_i + \frac{\partial {}^{t^*}_{0}f_i}{\partial {}^{t}\gamma}\Delta\gamma \\
{}^{t^*+\Delta t}_{0}T_i &\approx {}^{t^*}_{0}T_i + \frac{\partial {}^{t^*}_{0}T_i}{\partial {}^{t}\gamma}\Delta\gamma \\
{}^{t^*+\Delta t}_{0}q_i &\approx {}^{t^*}q_i + \frac{\partial {}^{t^*}q_i}{\partial {}^{t}u_j}u_j + \frac{\partial {}^{t^*}q_i}{\partial {}^{t}\dot{u}_j}\dot{u}_j + \frac{\partial {}^{t^*}q_i}{\partial {}^{t}\gamma}\Delta\gamma \\
{}^{t^*+\Delta t}_{0}Q_j &\approx {}^{t^*}_{0}Q_i + \frac{\partial {}^{t^*}_{0}Q_i}{\partial {}^{t}u_j}u_j + \frac{\partial {}^{t^*}_{0}Q_i}{\partial {}^{t}\dot{u}_j}\dot{u}_j + \frac{\partial {}^{t^*}_{0}Q_i}{\partial {}^{t}\gamma}\Delta\gamma,
\end{aligned}
\tag{52}
$$

where

$$\Delta\gamma = {}^{t^*+\Delta t}\gamma - {}^{t^*}\gamma.$$

Thus, eqn (50) becomes

$$
\begin{aligned}
&\Bigg[\int_{t_0}^{t_1}\int_{{}^0V}\{-{}^0\rho\,{}^{t^*}\ddot{u}_i u_i^a - {}^{t^*}_{0}S_{ij}\,{}^{t^*}_{0}\epsilon_{ij}^a + {}^0\rho\,{}^{t^*}_{0}f_i u_i^a \\
&+ {}^0\rho\,{}^{t^*}q_i u_i^a\}\,{}^0\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\{{}^{t^*}_{0}T_i u_i^a \\
&+ {}^{t^*}_{0}x_{i,j}\,{}^{t^*}_{0}Q_j u_i^a\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] + \Bigg[\int_{t_0}^{t_1}\int_{{}^0V}\Big\{-{}^0\rho\ddot{u}_i u_i^a \\
&-({}^{t^*}_{0}S_{ij}\,{}_0\eta_{ij}(u_k,u_k^a) + {}_0S_{ij}\,{}^{t^*}_{0}\epsilon_{ij}^a) + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}u_j}u_j u_i^a \\
&+ {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\dot{u}_j}\dot{u}_j u_i^a\Big\}\,{}^0\mathrm{d}V\,\mathrm{d}t + \int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial u_i}{\partial {}^0x_j}{}^{t^*}_{0}Q_j u_i^a \\
&+ {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}u_k}u_k u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\dot{u}_k}\dot{u}_k u_i^a\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] \\
&+\Bigg[\int_{t_0}^{t_1}\int_{{}^0V} -{}_0S_{ij}\,{}_0\eta_{ij}(u_k,u_k^a)\,{}^0\mathrm{d}V\,\mathrm{d}t \\
&+\int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial u_i}{\partial {}^0x_j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}u_k}u_k u_i^a \\
&+\frac{\partial u_i}{\partial {}^0x_j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\dot{u}_k}\dot{u}_k u_i^a\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t\Bigg] \\
&+\Bigg[\int_{t_0}^{t_1}\Delta\gamma\Big\{\int_{{}^0V}\Big({}^0\rho\frac{\partial {}^{t^*}_{0}f_i}{\partial {}^{t}\gamma}u_i^a + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}V \\
&+\int_{{}^0\Gamma_T}\Big(\frac{\partial {}^{t^*}_{0}Q_i}{\partial {}^{t}\gamma}u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}\Gamma_T\Big\}\mathrm{d}t\Bigg] = 0.
\end{aligned}
\tag{53}
$$

Because of the compatibility between $(u_i^a, {}^{t^*}_{0}\epsilon_{ij}^a)$ and $(\delta^{t^*}u_i, \delta^{t^*}_{0}\epsilon_{ij})$, the first bracket is equivalent to the equation of motion for the critical state t^* and, therefore, vanishes. The third bracket will be neglected because of its higher order. The second bracket is used to set up the adjoint equation from which the adjoint displacement u_i^a will be compound, i.e.

$$
\begin{aligned}
&\int_{t_0}^{t_1}\int_{{}^0V}\Big\{-{}^0\rho\ddot{u}_i u_i^a - ({}^{t^*}_{0}S_{ij}\,{}_0\eta_{ij}(u_k,u_k^a) + {}_0S_{ij}\,{}^{t^*}_{0}\epsilon_{ij}^a) \\
&+{}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}u_j}u_j u_i^a + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\dot{u}_j}\dot{u}_j u_i^a\Big\}\,{}^0\mathrm{d}V\,\mathrm{d}t \\
&+\int_{t_0}^{t_1}\int_{{}^0\Gamma_T}\Big\{\frac{\partial u_i}{\partial {}^0x_j}{}^{t^*}_{0}Q_j u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}u_k}u_k u_i^a \\
&+{}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\dot{u}_k}\dot{u}_k u_i^a\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t = 0.
\end{aligned}
\tag{54}
$$

Thus, eqn (53) reduces to

$$
\begin{aligned}
&\int_{t_0}^{t_1}\Delta\gamma\Big\{\int_{{}^0V}\Big({}^0\rho\frac{\partial {}^{t^*}_{0}f_i}{\partial {}^{t}\gamma}u_i^a + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}V \\
&+\int_{{}^0\Gamma_T}\Big(\frac{\partial {}^{t^*}_{0}T_i}{\partial {}^{t}\gamma}u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}\Gamma_T\Big\}\mathrm{d}t = 0.
\end{aligned}
\tag{55}
$$

Next, a procedure similar to that in Sec. 3.2.3 is used, i.e. $u_i = \hat{u}_i e^{\omega t}$, where $\hat{u}_i$ are amplitudes of u_i. u_i, $\hat{u}_i$, and u_i^a are all kinematically admissible because they vanish along displacement prescribed boundaries. Since limits for the time integral are arbitrary and the time exponential is not zero, the following condition is obtained from eqns (54) and (55):

$$
\begin{aligned}
&\int_{{}^0V}\Big\{-{}^0\rho\hat{u}_i u_i^a\omega^2 - \{{}^{t^*}_{0}S_{ij}\,{}_0\eta_{ij}(\hat{u}_k,u_k^a) \\
&+{}^{t^*}\phi_{ijkl}\,{}_0e_{kl}({}^{t^*}u,\hat{u})\,{}^{t^*}_{0}\epsilon_{ij}^a\} \\
&+{}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}u_j}\hat{u}_j u_i^a + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\dot{u}_j}\hat{u}_j u_i^a\omega\Big\}\,{}^0\mathrm{d}V\,\mathrm{d}t \\
&+\int_{{}^0\Gamma_T}\Big\{\frac{\partial \hat{u}_i}{\partial {}^0x_i}{}^{t^*}_{0}Q_j u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}u_k}\hat{u}_k u_i^a \\
&+{}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\dot{u}_k}\hat{u}_k u_i^a\omega\Big\}\,{}^0\mathrm{d}\Gamma_T\,\mathrm{d}t = 0
\end{aligned}
\tag{56}
$$

and

$$
\begin{aligned}
&\Delta\gamma\Big\{\int_{{}^0V}\Big({}^0\rho\frac{\partial {}^{t^*}_{0}f_i}{\partial {}^{t}\gamma}u_i^a + {}^0\rho\frac{\partial {}^{t^*}q_i}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}V \\
&+\int_{{}^0\Gamma_T}\Big(\frac{\partial {}^{t^*}_{0}T_i}{\partial {}^{t}\gamma}u_i^a + {}^{t^*}_{0}x_{i,j}\frac{\partial {}^{t^*}_{0}Q_j}{\partial {}^{t}\gamma}u_i^a\Big){}^0\mathrm{d}\Gamma_T\Big\} = 0.
\end{aligned}
\tag{57}
$$

Equation (56) yields a quadratic eigenvalue problem with the same characteristic equation as the

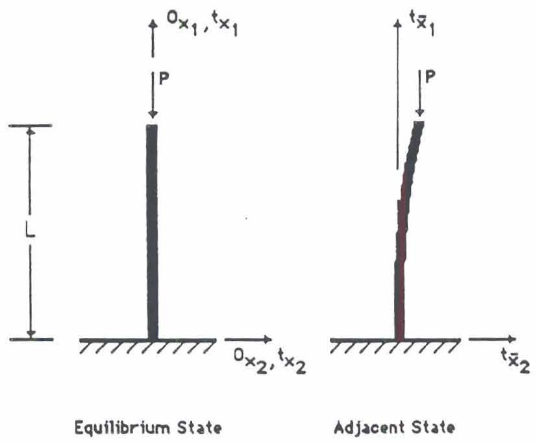

Fig. 3. Beam–column under tip force.

stability criterion (46). With the value of ω at the critical point, u_i^a can be recognized as the left eigenfunction of the stability criterion and $\hat{u}_i$ the right eigenfunction. Furthermore, $\hat{u}_i$ and $\tilde{u}_i$ are, in fact, identical.

The study of (57) leads to the distinction between the limit point and the bifurcation point instability which can be summarized as follows:

1. If the terms in the brackets of eqn (57) vanish, $\Delta\gamma$ does not have to be zero. This indicates that, at the critical point, the load increment exists for a perfect structure and the system moves in the direction of unstable branch of motion. The point satisfying this behavior is called the bifurcation point.
2. If the summation of the terms in the brackets of eqn (57) does not vanish, $\Delta\gamma$ must be zero. This means that the load cannot be inceased in the neighborhood of the critical point. Therefore, the point encountered here is the limit point.

Note that the criterion in eqn (57) can be used only after the critical state and the corresponding load have been determined.

4. COMPUTATIONAL PROCEDURE

In this section, the general stability criterion in eqn (45) is used to set up the characteristic equation for both conservative and nonconservative problems. Though the examples are fairly simple, they give valuable insight into the properties of ω, and computational procedures for more complex problems.

4.1. *Beam–column under tip force*

The column under the conservative tip force is investigated. The equilibrium state and the adjacent equilibrium state together with the reference coordinates are shown in Fig. 3. The assumptions of a slender beam are used here. Therefore the deformed coordinates and the displacements are

$$\begin{Bmatrix} {}^tx_1 \\ {}^tx_2 \end{Bmatrix} = \begin{Bmatrix} {}^0x_1 + {}^tu_{1a} - {}^0x_2 \dfrac{\partial\, {}^tu_2}{\partial\, {}^0x_1} \\ {}^0x_2 + {}^tu_2 \end{Bmatrix}$$

$$\begin{Bmatrix} {}^tu_1 \\ {}^tu_2 \end{Bmatrix} = \begin{Bmatrix} {}^tu_{1a} - {}^0x_2 \dfrac{\partial\, {}^tu_2}{\partial\, {}^0x_1} \\ {}^tu_2 \end{Bmatrix}, \tag{58}$$

where ${}^tu_{1a}$ is the displacement due to the axial load. Similarly, one obtains

$$\begin{Bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{Bmatrix} = \begin{Bmatrix} \tilde{u}_{1a} - {}^0x_2 \dfrac{\partial \tilde{u}_2}{\partial\, {}^0x_1} \\ \tilde{u}_2 \end{Bmatrix}$$

$$\begin{Bmatrix} \delta\tilde{u}_1 \\ \delta\tilde{u}_2 \end{Bmatrix} = \begin{Bmatrix} \delta\tilde{u}_{1a} - {}^0x_2 \dfrac{\delta\partial\tilde{u}_2}{\partial\, {}^0x_1} \\ \delta\tilde{u}_2 \end{Bmatrix}. \tag{59}$$

${}_0e_{ij}({}^t\mathbf{u}, \tilde{\mathbf{u}})$, ${}_0e_{ij}({}^t\mathbf{u}, \delta\tilde{\mathbf{u}})$ and ${}_0\eta_{ij}(\tilde{u}_k, \delta\tilde{u}_k)$ can be expressed as

$${}_0e_{ij}({}^t\mathbf{u}, \tilde{\mathbf{u}}) = \begin{bmatrix} \dfrac{\partial\tilde{u}_{1a}}{\partial\, {}^0x_1} - {}^0x_2 \dfrac{\partial^2\tilde{u}_2}{\partial\, {}^0x_1^2} & 0 \\ 0 & 0 \end{bmatrix} + \text{h.o.t.}$$

$${}_0e_{ij}({}^t\mathbf{u}, \delta\tilde{\mathbf{u}}) = \begin{bmatrix} \dfrac{\partial\delta\tilde{u}_{1a}}{\partial\, {}^0x_1} - {}^0x_2 \dfrac{\partial^2\delta\tilde{u}_2}{\partial\, {}^0x_1^2} & 0 \\ 0 & 0 \end{bmatrix} + \text{h.o.t.}$$

$${}_0\eta_{ij}(\tilde{u}_k, \delta\tilde{u}_k) = \begin{bmatrix} {}_0\eta_{11} & {}_0\eta_{12} \\ {}_0\eta_{21} & {}_0\eta_{22} \end{bmatrix}, \tag{60}$$

where

$${}_0\eta_{11} = \tilde{u}_{1a,1}\delta\tilde{u}_{1a,1} - {}^0x_2(\tilde{u}_{2,11}\delta\tilde{u}_{1a,1} + \tilde{u}_{1a,1}\delta\tilde{u}_{2,11})$$

$$+ {}^0x_2{}^2\tilde{u}_{2,11}\delta\tilde{u}_{2,11} + \tilde{u}_{2,1}\delta\tilde{u}_{2,1}$$

$${}_0\eta_{12} = {}_0\eta_{21} = \tfrac{1}{2}\{-\tilde{u}_{1a,1}\delta\tilde{u}_{2,1} + {}^0x_2\tilde{u}_{2,11}\delta\tilde{u}_{2,1}$$

$$- \tilde{u}_{2,1}\delta\tilde{u}_{1a,1} + {}^0x_2\tilde{u}_{2,1}\delta\tilde{u}_{2,11}\}$$

$${}_0\eta_{22} = \tilde{u}_{2,1}\delta\tilde{u}_{2,1}\,.$$

Also, the state of stress in the equilibrium state t with linear elastic material is

$$[{}_0^tS_{ij}] = \begin{bmatrix} -\dfrac{P}{A} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E(u_{1a,1} - {}^0x_2u_{2,11}) & 0 \\ 0 & 0 \end{bmatrix}. \tag{61}$$

Substituting eqns (60) and (61) into eqn (46), one obtains the equation for the stability criterion as

$$\int_0^L \{-{}^0\rho A\tilde{u}_{1a}\delta\tilde{u}_{1a}\omega^2 - EA\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1} + P\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1}\}\,{}^0\mathrm{d}x_1 + \int_0^L\left\{-{}^0\rho I\tilde{u}_{2,1}\delta\tilde{u}_{2,1}\omega^2 - {}^0\rho A\tilde{u}_2\delta\tilde{u}_2\omega^2 - EI\tilde{u}_{2,11}\delta\tilde{u}_{2,11} + P\tilde{u}_{2,1}\delta\tilde{u}_{2,1} + \frac{PI}{A}\tilde{u}_{2,11}\delta\tilde{u}_{2,11}\right\}{}^0\mathrm{d}x_1 = 0. \quad (62)$$

Using integration by parts, eqn (62) yields two uncouple differential equations with separate boundary conditions on variables $\tilde{u}_{1a}$ and $\tilde{u}_2$. Those differential equations and their associated boundary conditions can be combined to obtain two conditions at the critical point as

$$\int_0^L \{-{}^0\rho A\tilde{u}_{1a}\delta\tilde{u}_{1a}\omega^2 - EA\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1} + P\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1}\}\,{}^0\mathrm{d}x_1 = 0 \quad (63)$$

and

$$\int_0^L\left\{-{}^0\rho I\tilde{u}_{2,1}\delta\tilde{u}_{2,1}\omega^2 - {}^0\rho A\tilde{u}_2\delta\tilde{u}_2\omega^2 - EI\tilde{u}_{2,11}\delta\tilde{u}_{2,11} + P\tilde{u}_{2,1}\delta\tilde{u}_{2,1} + \frac{PI}{A}\tilde{u}_{2,11}\delta\tilde{u}_{2,11}\right\}{}^0\mathrm{d}x_1 = 0. \quad (64)$$

To be consistent with the equilibrium equation the last term in eqn (63) is neglected. As a result eqn (63) yields the solution of $\tilde{u}_1^a = 0$ for any value of load parameter with elastic material model. Therefore, only eqn (63) becomes the stability criterion for this problem. If the span–depth ratio of the beam is relatively large—namely the deformation due to shear and initial resistance to rotational acceleration of the beam cross section are small, the rotational effects may be neglected. Therefore, the following terms vanish:

$$\int_0^L -{}^0\rho I\tilde{u}_{2,1}\delta\tilde{u}_{2,1}\omega^2\,{}^0\mathrm{d}x_1 \to 0$$

$$\int_0^L \frac{PI}{A}\tilde{u}_{2,11}\delta\tilde{u}_{2,11}\,{}^0\mathrm{d}x_1 \to 0. \quad (65)$$

Finally, the stability condition in eqn (63) becomes

$$\int_0^L \{-{}^0\rho A\tilde{u}_2\delta\tilde{u}_2\omega^2 - EI\tilde{u}_{2,11}\delta\tilde{u}_{2,11} + P\tilde{u}_{2,1}\delta\tilde{u}_{2,1}\}{}^0\mathrm{d}x_1 = 0. \quad (66)$$

Equation (66) can be discretized by any standard finite element method and yields the frequency eigenvalue problem of the following form.

$$[M\omega^2 + K - K_G]\tilde{U} = 0, \quad (67)$$

where M, K, and K_G are mass, linear stiffness and geometric stiffness matrices, respectively, and $\tilde{U}$ is mode shape of the buckled state corresponding to the eigenvalue ω^2.

Using a two-element model as shown in Fig. 4 and standard beam finite elements, the matrices M, K, and K_G are calculated as

$$M = \frac{{}^0\rho Al}{420}\begin{bmatrix} 312 & 54 & 0 & -13l \\ 54 & 156 & 13l & -22l \\ 0 & 13l & 8l^2 & -3l^2 \\ -13l & -22l & -3l^2 & 4l^2 \end{bmatrix}$$

$$K = \frac{EI}{l^3}\begin{bmatrix} 24 & -12 & 0 & 6l \\ -12 & 12 & -6l & -6l \\ 0 & -6l & 8l^2 & 2l^2 \\ 6l & -6l & 2l^2 & 4l^2 \end{bmatrix}$$

$$K_G = \frac{-P}{30l}\begin{bmatrix} 72 & -36 & 0 & 3l \\ -36 & 36 & -3l & -3l \\ 0 & -3l & 8l^2 & -l^2 \\ 3l & -3l & -l^2 & 4l^2 \end{bmatrix}, \quad (68)$$

where $l = L/2$. Note that K_G explicitly depends on the magnitude of the load, P. Thus, the eigenvalue problem (67) can be solved for ω^2 at a given value of P. As P is increased incrementally, the associated eigenvalue ω^2 is computed and monitored. The numerical solution of the eigenvalue problem is performed by the subroutine *GVCCG* in the *IMSL* subroutine library. The largest eigenvalue ω^2 changes sign from negative to positive when P is approximately $\pi^2/4\ EI/L^2$. The load–eigenvalue relation is shown in Fig. 5.

It is important to note that since the problem is conservative, the load–frequency eigenvalue problem (67) can be solved directly for the critical load by

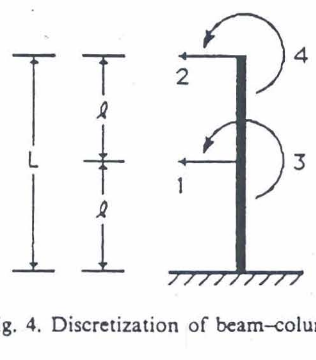

Fig. 4. Discretization of beam–column.

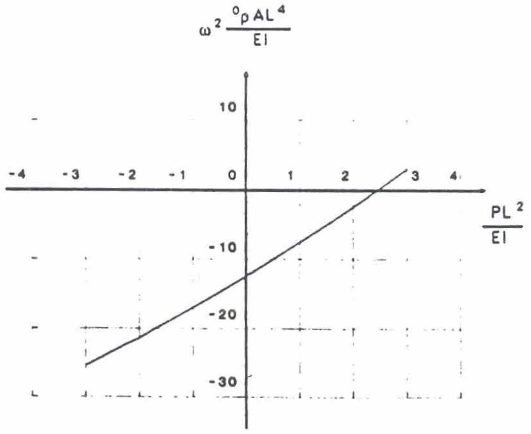

Fig. 5. Load vs largest eigenvalue variation for beam–column under tip force.

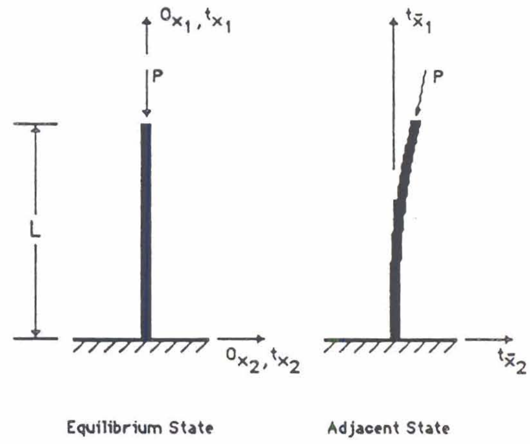

Fig. 6. Beam–column under follower force.

setting $\omega^2 = 0$ (refer to Sec. 3.2.3). The result is identical to that obtained using the stability criterion for conservative systems given in eqn (30). In this case, the eigenvalue problem will be only in terms of the critical load parameter that needs to be solved once. This is the usual procedure for computing the critical load for conservative systems. The procedure in the foregoing paragraph is used merely to demonstrate the use of general stability criterion given in eqn (46).

To demonstrate that the current problem is that of bifurcation instability rather than limit point instability the criterion given in eqn (57) is used. For the problem, the magnitude of the applied load at the column tip is selected as the load parameter. Therefore,

$$\frac{\partial {}^{t*}_0 T_1}{\partial {}^t\gamma} = -\frac{1}{A} \tag{69}$$

while other terms are zero in eqn (57). Also, because of problem symmetry, the adjoint displacements u_1^a are identical to $\tilde{u}_i$. Thus, eqn (57) becomes

$$\begin{aligned}\Delta\gamma \int_{{}^0\Gamma_T} \frac{\partial {}^{t*}_0 T_i}{\partial {}^t\gamma} u_1^{a\,0}\mathrm{d}\Gamma_T &= \Delta\gamma \int_A -\frac{1}{A}\left(\tilde{u}_{1a} - x_2 \frac{\partial \tilde{u}_2}{\partial {}^0x_1}\right) \mathrm{d}A\,|_{x_1=L} \\ &= \Delta\gamma\left(-\frac{1}{A}\tilde{u}_{1a}\right)\Big|_{x_1=L}.\end{aligned} \tag{70}$$

Since $\tilde{u}_{1a}$ has been determined to be zero with the elastic material model, $\Delta\gamma$ need not be zero in the above equation. As a result, it is a bifurcation point instability problem.

4.2. *Beam–column with follow tip force*

The problem of the previous subsection is now reconsidered under the follower tip force, which is a nonconservative force. The tip force remains tangential to the column's tip at all times. The problem reference axis is given in Fig. 6.

Similar to the previous example, eqns (58)–(61) hold. However, in addition to the terms in eqn (61), there is a non-zero boundary integral term coming from the effect of the nonconservative force

$$\int_{{}^0\Gamma_T} \frac{\partial \tilde{u}_i}{\partial {}^0x_j} {}^t_0Q_j \delta\tilde{u}_i\, {}^0\mathrm{d}\Gamma_T = -[P\tilde{u}_{1a,1}\delta\tilde{u}_{1a} + P\tilde{u}_{2,1}\delta\tilde{u}_2]_{{}^0x_1=L}. \tag{71}$$

Thus, the stability criterion of eqn (46), after ignoring the rotational terms, gives two necessary conditions at the critical point

$$\int_0^L \{-{}^0\rho A\tilde{u}_{1a}\delta\tilde{u}_{1a}\omega^2 - EA\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1} + P\tilde{u}_{1a,1}\delta\tilde{u}_{1a,1}\}\, {}^0\mathrm{d}x_1 - P\tilde{u}_{1a,1}\delta\tilde{u}_{1a}|_{{}^0x_1=L} = 0 \tag{72}$$

$$\int_0^L \{-{}^0\rho A\tilde{u}_2\delta\tilde{u}_2\omega^2 - EI\tilde{u}_{2,11}\delta\tilde{u}_{2,11} + P\tilde{u}_{2,1}\delta\tilde{u}_{2,1}\}\, {}^0\mathrm{d}x_1 - P\tilde{u}_{2,1}\delta\tilde{u}_2|_{{}^0x_1=L} = 0. \tag{73}$$

As in the previous example, the last terms in eqn (72) are neglected. Thus, only eqn (73) governs the critical state. Equation (70) can be discretized to yield the load–frequency eigenvalue problem as

$$[M\omega^2 + K - K_G + \bar{K}_G]\tilde{U} = 0, \tag{74}$$

where M, K, and K_G, using the two-beam-element model, are identical to the previous problem as given in eqn (68). The term $\bar{K}_G$ is the additional geometric

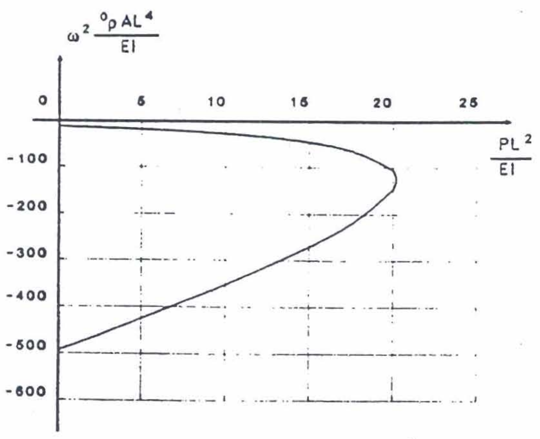

Fig. 7. Variation of two largest eigenvalues vs load P for beam–column under follower force.

stiffness due to the nonconservative force, and for the two-beam-element model, it is given as

$$\tilde{K}_G = \frac{-P}{30l}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{75}$$

The presence of the $\tilde{K}_G$ makes the eigenproblem unsymmetric and, therefore, ω^2 may not be zero at the critical state. Since $\tilde{K}_G$ also depends on the load parameter P explicitly as in K_G, a numerical procedure similar to that used in previous example is repeated. As the load P is increased, however, two of the eigenvalues ω^2 become complex before a positive value of ω^2 is encountered. The transition occurs when the load is about $20.19\ EI/L^2$, which yields the buckling load for the column. Bolotin's analysis [32] also gave the buckling load of $20.19\ EI/L^2$ for this problem while Beck [54] gave the critical load as $20.05\ EI/L^2$. Deineko and Leonov [55] found the buckling load to be around $2\pi^2 EI/L^2$

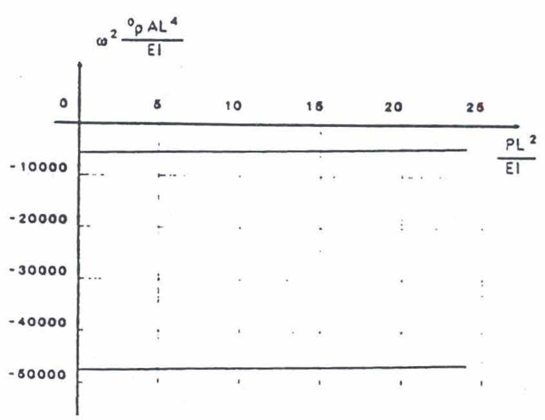

Fig. 8. Variation of two smallest eigenvalues vs load P for beam–column under follower force.

($\approx 19.77\ EI/L^2$). The characteristic of all four ω^2 are shown in Figs 7 and 8.

Compared to the previous problem, one sees that the nature of the problem is changed when the load becomes nonconservative. The frequency, instead of moving from a stable to an unstable region through the origin, now moves out of the stable region at other points along the β axis. Also, the largest eigenvalue is not necessarily the one to dominate the problem characteristic.

To show that this is also a problem of bifurcation instability only the axial component of the left eigenfunction of the stability criterion is required. The coefficient of $\Delta\gamma$ in eqn (57) is given as

$$\int_{{}^0\Gamma_T} {}^{t^*}_0 x_{i,j} \frac{\partial_0^{t^*} Q_j}{\partial^t \gamma} u_i^a \,{}^0 d\Gamma_T = \int -\frac{1}{A}\left(u^a_{1,*} - x_2 \frac{\partial u_2^a}{\partial^0 x_1}\right) + \frac{\partial^{t^*} u_2}{\partial^0 x_1}\left(-\frac{1}{A}\right) u_2^a \,{}^0 d\Gamma_T + \text{h.o.t.}$$

$$= -u^a_{1a} - \frac{\partial^{t^*} u_2}{\partial^0 x_1} u_2^a, \tag{76}$$

where u^a_{1a} is the axial adjoint displacement. Since eqn (72) is symmetric after neglecting the last two terms, u^a_{1*} is identical to $\tilde{u}_{1a}$ and is zero. Also, since the transverse displacement t_{u_2} does not occur, $\partial^{t^*} u_2/\partial^0 x_1 = 0$. Thus, eqn (76) vanishes, and it is again a bifurcation instability problem.

5. DISCUSSION AND CONCLUSIONS

In this paper, theories for the stability of mechanical and structural systems are presented. The Liapunov second method for the study of the stability of dynamical systems is described, and an extension of the method to conservative systems is provided. The stability criteria for both the conservative and nonconservative structural systems are then developed using continuum formulations and the Total Lagrangian concept. The criteria can be applied to such practical structural systems as trusses, beams, frames, plates, and shells. The continuum expressions can be discretized using isoparametric finite element procedures, so that the theory can be applied to complex structures. Additional work is usually required before the final computer implementable equations are obtained. Furthermore, since the geometric and material nonlinearities (with the incremental constitutive material law) are included in the formulations, the criteria can be applied to highly nonlinear structural problems. Even though the criteria are formulated based on the Total Lagrangian formulations, they can be extended for the Updated Lagrangian formulation.

Though, only the linear material model is used in both examples, the stability criteria are applicable to systems with nonlinear material behavior. However, appropriate material constitutive models need to be used. For plasticity problems, the loading or unloading material modulus tensor needs to be used depending on whether the stress is increasing or decreasing at a point of the continuum.

In addition to the stability criteria, a variational expression is developed to distinguish between the limit point and bifurcation point instability. It is important to identify the type of instability when analysis beyond the critical point or design sensitivity analysis of the critical load is needed.

From the study, the following conclusions are drawn.

(1) Characteristics of the Liapunov functions determine the stability of the dynamical systems.
(2) Liapunov's direct method leads to the criterion of the existence of a minimum for the total potential energy for conservative systems in motion. This is called Euler's method. The method is extended to nonlinear problems.
(3) The stability criterion for conservative systems is simply a special case of the criterion for nonconservative systems.
(4) In the stability investigation of systems with nonconservative applied forces, whether in static equilibrium or in motion, the dynamic method must be employed.
(5) The general stability investigation is equivalent to the study of the natural frequencies of the systems in the deformed state.

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