# 2-1 STRAIN ENERGY OF BEAMS, COLUMNS AND SHAFTS

## STRAIN ENERGY DUE TO BENDING

- Elastic material: internal forces are conservative in the kinetic sense.
- Straight beam subjected to bending about two axes and load along its axis; consider principal axes of inertia.

$$\int \xi \mathrm{dA} = 0, \quad \int \eta \mathrm{dA} = 0, \quad \int \eta\xi \mathrm{dA} = 0$$

- Assumptions: plane sections remain plane and normal to the centroidal axis.
- Derive expressions for moments along two axes using vector product.

$$s = \mathrm{a}x + \mathrm{b}h + \mathrm{c}$$

$$\mathbf{M} = \mathbf{r}\ ´\ \mathbf{F}, \quad \mathbf{r} = \xi \mathbf{e}_1 + \eta \mathbf{e}_2, \ \mathbf{F} = \int \sigma \mathrm{dA}, \ \mathrm{s} = \sigma \mathbf{e}_3$$

$$\mathrm{M}_x = \int s h \mathrm{dA}; \quad \mathrm{M}_h = -\int s x \mathrm{dA}; \quad \mathrm{N} = \int s \ \mathrm{dA}$$

- $s = -\frac{\mathrm{M}_h x}{\mathrm{I}_h} + \frac{\mathrm{M}_x h}{\mathrm{I}_x} + \frac{\mathrm{N}}{\mathrm{A}}$

- Derive strain energy density expression: virtual work of the internal force on a small differential volume element dsdA is given as

  $w'_i = \int(\sigma dAds)de$ = - 1/2 $E\varepsilon^2 dAds$ = -1/2 $\sigma\varepsilon$ dAds

  Strain energy of small element, dU = 1/2 $\sigma\varepsilon$ dAds

  Strain energy density, $U_0$ = 1/2 $\sigma\varepsilon$

- Strain energy expression for the beam; $\int U_0 dV = \frac{1}{2}\int se\ dV$

- Strain energy is sum of three strain energies

$$U = \frac{1}{2E}\int_0^L \left( \frac{M_x^2}{I_x} + \frac{M_h^2}{I_h} + \frac{N^2}{A} \right) ds$$

- Moment curvature relationship: $M = EI/R$

- Strain energy for bending in one plane

$$U = \int_0^L \frac{EIds}{2R^2} = \frac{1}{2}\int_0^L EI\left[(x'')^2 + (y'')^2\right]ds = \frac{1}{2}\int_0^L EI(y'')^2 dx$$

## STRAIN ENERGY DUE TO SHEAR

- Shear stress is approximated by the elementary formula

  $t = SQ/Ib$

- Strain energy density due to shear deformation: $t^2/2G$

- Strain energy due to shear deformation: $U_s = \int_0^L \frac{kS^2}{2GA} dx$ ;

$$k = \frac{A}{I^2} \int \frac{Q^2}{b} d?$$

- $U_s = \frac{1}{2} \int_0^L \beta S dx$; β = slope due to shear deformation, $\beta = \frac{kS}{GA}$
- y includes deflection due to shear; y' is the total slope
- Slope due to bending only = (y' - slope due to shear); similarly curvature

$$U_b = \int_0^L EI[y'' - \beta']^2 dx$$

- The problem is to determine y and β to minimize the total potential energy among the functions that satisfy the end conditions and continuity requirements.
- For clamped end: y = 0, y' = β, M = EI(y" - β')
- For pinned end: y = 0, M = 0 gives y" = β'
- For free end: M = 0 gives y" = β'

  S = 0 (dM/dx = 0) gives y"' = β''
- At the point where a point load is applied, y' and β are generally discontinuous, but (y' - β) and (y" - β') are

continuous, since the bending moment is continuous. $(y''' - \beta'')$ being proportional to shear is discontinuous.

## STRAIN ENERGY DUE TO TORSION

- Derive an expression for strain energy due to torsion
- Expression for torque: $T = GJ\theta/L$
- Strain energy due to torsional deformation: $U_T = \frac{GJq^2}{2L}$
- Total strain energy = sum of all strain energies