2-1 STRAIN ENERGY OF BEAMS, COLUMNS AND SHAFTS

STRAIN ENERGY DUE TO BENDING

➢ Elastic material: internal forces are conservative in the kinetic sense.

➢ Straight beam subjected to bending about two axes and load along its axis; consider principal axes of inertia.

\[ \int \xi dA = 0, \quad \int \eta dA = 0, \quad \int \eta \xi dA = 0 \]

➢ Assumptions: plane sections remain plane and normal to the centroidal axis.

➢ Derive expressions for moments along two axes using vector product.

\[ s = a? + b? + c \]

\[ M = r \times F, \quad r = \xi e_1 + \eta e_2, \quad F = \int \sigma dA, \quad \sigma = \sigma e_3 \]

\[ M_? = \int s? dA; \quad M_? = -\int s? dA; \quad N = \int s dA \]

➢ \[ s = -\frac{M_?}{I_?} + \frac{M_?}{I_?} + \frac{N}{A} \]
Derive strain energy density expression: virtual work of the internal force on a small differential element dsdA is given as

\[ w_i' = \int (\sigma dA) d\varepsilon = -1/2 E \varepsilon^2 dA \]

Strain energy of small element, dU = 1/2 \( \sigma \varepsilon \) dA

Strain energy density, \( U_0 = 1/2 \sigma \varepsilon \)

Strain energy expression for the beam: \( \frac{1}{2} \int U_0 dV = \frac{1}{2} \int \sigma \varepsilon dV \)

Strain energy is sum of three strain energies

\[ U = \frac{1}{2E_0} \int \left( \frac{M_x^2}{I_x} + \frac{M_y^2}{I_y} + \frac{N^2}{A} \right) ds \]

Moment curvature relationship: \( M = EI/R \)

Strain energy for bending in one plane

\[ U = \int_0^L \frac{EI ds}{2R^2} = \frac{1}{2} \int_0^L EI [x''^2 + (y'')^2] ds = \frac{1}{2} \int_0^L EI(y'')^2 dx \]

**STRAIN ENERGY DUE TO SHEAR**

Shear stress is approximated by the elementary formula

\[ t = SQ/Ib \]

Strain energy density due to shear deformation: \( t^2/2G \)
Strain energy due to shear deformation: \[ U_s = \int_0^L \frac{?S^2}{2GA} \, dx; \]

\[ ? = \frac{A}{I^2} \int \frac{Q^2}{b} \, d? \]

\[ U_s = \frac{1}{2} \int_0^L \beta S \, dx; \quad \beta = \text{slope due to shear deformation}, \quad \beta = \frac{?S}{GA} \]

\[ y \text{ includes deflection due to shear; } y' \text{ is the total slope} \]

Slope due to bending only = \( y' - \text{slope due to shear} \); similarly curvature

\[ U_b = \int_0^L EI[y'' - \beta']^2 \, dx \]

The problem is to determine \( y \) and \( \beta \) to minimize the total potential energy among the functions that satisfy the end conditions and continuity requirements.

For clamped end: \( y = 0, y' = \beta, M = EI(y'' - \beta') \)

For pinned end: \( y = 0, M = 0 \) gives \( y'' = \beta' \)

For free end: \( M = 0 \) gives \( y'' = \beta' \)

\[ S = 0 \text{ (} \frac{dM}{dx} = 0 \text{) gives } y''' = \beta'' \]

At the point where a point load is applied, \( y' \) and \( \beta \) are generally discontinuous, but \( (y' - \beta) \) and \( (y'' - \beta') \) are
continuous, since the bending moment is continuous. \((y''') - 
\beta''\) being proportional to shear is discontinuous.

**STRAIN ENERGY DUE TO TORSION**

- Derive an expression for strain energy due to torsion
- Expression for torque: \(T = GJ\theta/L\)
- Strain energy due to torsional deformation: \(U_T = \frac{GJ\theta^2}{2L}\)
- Total strain energy = sum of all strain energies