A Fast & Accurate Non-Iterative Algorithm for Regularized Non-Cartesian MRI

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***Abstract*—We introduce a novel algorithm for regularized reconstruction of non-Cartesian MRI data. The proposed non-iterative scheme closely approximates the Tikhonov regularized least squares method, but provides a significant speed up over standard implementation based on iterative conjugate gradient algorithm. This computational complexity of the proposed scheme is comparable to that of gridding. However, the proposed scheme is significantly more robust to undersampling and measurement noise. Numerical simulations clearly demonstrate the advantages of the proposed algorithm over traditional schemes. The proposed algorithm may be very useful in dynamic and functional MRI applications, where the fast reconstruction of several undersampled images is required.**

***Index Terms*—Magnetic Resonance Imaging, Sparse Reconstruction, Orthogonal Matching Pursuits, Thikanov Regularization.**

I. Introduction

The reconstruction of images from its non-uniform Fourier samples arise in many areas including tomography [1], magnetic resonance imaging [2], and synthetic aperture radar [3]. This problem is especially important in MRI, since many of the modern pulse sequences exploit non-Cartesian sampling schemes to significantly speed up dynamic and functional MRI acquisitions.

The widely used non-Cartesian MRI reconstruction scheme is gridding. Gridding schemes interpolate the non-uniform Fourier samples, weighted by the density compensation kernel, on to a uniform grid. The image is recovered from this data using a simple IFFT, followed by weighting to compensate for the interpolation. The simplicity and the computational efficiency of gridding is the main reason for its widespread use. Several authors have addressed the optimization of density compensation functions, interpolators, and post-compensation weights to improve the quality of gridding [4], [5], [6].

It is now well accepted that regularized least-squares schemes are considerably more accurate and robust to noise and under-sampling than gridding reconstructions. However, the main limitation of this scheme is its computational complexity, which is especially a problem when dealing with large multi-dimensional datasets. For example, the reconstruction of a typical spiral f-MRI dataset involves the recovery of around 1000 images; the direct use of CG to reconstruct each image takes several hours to reconstruct the entire dataset.

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We propose a novel reconstruction algorithm to overcome the above mentioned problems. Specifically, we aim to achieve the accuracy of Tikhonov regularized method, but with almost the same computational complexity of gridding algorithms. The proposed scheme is a simple two-step algorithm, which involves (a) replacing each non-uniform Fourier sample by a weighted linear combination of a few non-uniform Fourier samples in its neighborhood (b), and an inverse non-uniform FFT (INUFFT) [7] scheme to recover the image from the weighed linear combination. The proposed scheme is a generalization of gridding, where the weighting of each non-uniform sample is replaced by a weighted linear combination.

The proposed scheme is conceptually similar to BURS and kSPA schemes [8], [5], introduced for un-regularized least-squares reconstruction. Both methods replace the non-uniform samples by a weighted linear combination of its N immediate neighbors. The BURS method compute the weights from the pseudo-inverse of an $N \times N$ sinc interpolation matrix. This is a crude approximation and is reported to give algorithms that are often sensitive to noise [9]. In contrast, kSPA poses the matrix approximation as a Frobenius norm minimization scheme similar to our approach. They decouple the approximation problem into several least squares problems, thus solving them efficiently.

The main difference of our approach with the above sparse approximation schemes are **(a)** the ability to account for Tikhonov regularization, and **(b)** the use of orthogonal matching pursuits (OMP) to select the optimal sparsity pattern. The use of Tikhonov regularization is crucial while dealing with under-sampled trajectories; it can significantly minimize aliasing artifacts over least square schemes and are much more robust to measurement noise. In BURS and kSPA each non-uniform sample is replaced by a weighted linear combination of its N immediate neighbors. In contrast, we determine the best non-uniform samples from the entire set. This significantly reduces the approximation error for a specified computational complexity. Hence, the proposed scheme can give solutions that are closer to the exact solutions.

II. Theory

A. Problem formulation

We restrict ourselves to considering two dimensional MRI case for simplicity. The extension to multiple dimensions is straightforward. The basic goal is to reconstruct an image $\rho(\mathbf{x}); \mathbf{x} \in \mathbb{R}^2$ from its non-uniform Fourier measurements

$$\hat{\rho}(\mathbf{k}_i) = \int_{\mathbf{x}\in\Omega} \rho(\mathbf{x}) \exp\left(j2\pi \mathbf{k}_i^T \mathbf{x}\right) d\mathbf{x};\ i = 0\ldots, L-1 \quad (1)$$

Assuming rectangular voxels, we rewrite this expression as the Fourier sum

$$\hat{\rho}(\mathbf{k}_i) = \sum_{j=0}^{M-1} \rho[\mathbf{m}] \exp\left(j2\pi \mathbf{k}_i^T \mathbf{m}_j\right);\ i = 0\ldots, L-1 \quad (2)$$

Collecting the measurements into an L dimensional vector, denoted by b, the above summation can be rewritten in the matrix form as

$$\mathbf{A}\mathbf{p} = \mathbf{b}, \quad (3)$$

where the entries of the $L \times M$ system matrix $\mathbf{A}$ are Fourier exponentials. The $M \times 1$ vector $\mathbf{p}$ is the collection of voxel values. $\mathbf{b}$ is the vector of measurements. In many cases of practical importance, the system is under-determined and the measurements are corrupted by noise.

The general approach to solving such underdetermined problems is the Tikhonov regularized least-squares reconstruction:

$$\mathbf{p}^* = \arg\min_{\mathbf{p}} \|\mathbf{A}\mathbf{p} - \mathbf{b}\|^2 + \lambda\|\mathbf{p}\|^2, \quad (4)$$

where λ is an arbitrary regularization parameter. The solution to this problem is given by

$$\mathbf{p}^* = \left(\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I}\right)^{-1}\mathbf{A}^T\mathbf{b} \quad (5)$$

$$= \mathbf{A}^T\left(\underbrace{\mathbf{A}\mathbf{A}^T + \lambda\mathbf{I}}_{\mathbf{P}}\right)^{-1}\mathbf{b} \quad (6)$$

The multiplication by $\mathbf{A}^T$ can be implemented efficiently as an INUFFT [7]. The operation $\mathbf{P}^{-1}\mathbf{b}$ involves replacing every non-uniform sample with a weighted linear combination of the set of non-uniform samples; the weights are specified by the corresponding row of $\mathbf{P}^{-1}$.

If $\mathbf{P}^{-1}$ is a full matrix, each non-uniform sample has to replaced by a linear combination of all the Fourier samples. In addition, the system matrix $\mathbf{A}$ is very large in many practical applications. This makes the evaluation, storage, and reconstruction using a full $\mathbf{P}$ matrix is almost impossible. The general practice in this setting is to solve for (4) using iterative conjugate gradient algorithms. While this significantly accelerates the computation over straightforward matrix operations, it is still too slow for large datasets, as discussed previously. In this context, we propose to approximate $\mathbf{P}^{-1}$ with a sparse matrix to obtain a fast and efficient algorithm.

B. Approximate inversion using a sparse matrix

The gridding scheme can be written in the matrix form as

$$\mathbf{p} = \mathbf{A}^T\mathbf{D}\mathbf{b}, \quad (7)$$

where $\mathbf{D}$ is a diagonal matrix with entries corresponding to the density compensation factors. Thus, the gridding scheme can be thought of as approximating $\mathbf{P}^{-1}$ with a diagonal matrix; the accuracy and robustness of gridding reconstruction is dependent on the quality of the approximation $\mathbf{D} \approx \mathbf{P}^{-1}$ [4].

To improve the approximation, we propose to approximate $\mathbf{P}$ by a sparse matrix $\mathbf{Q}$ in the least squares sense as

$$\mathbf{Q} = \arg\min_{\mathbf{Q};\mathbf{Q}\in\text{N-sparse}} \|\mathbf{Q}\mathbf{P} - \mathbf{I}\|_F^2 \quad (8)$$

Here, $\|\mathbf{P}\|_F$ is the Frobenius norm of $\mathbf{P}$. Instead of modeling $\mathbf{Q}$ as a diagonal matrix, we constrain each of the rows of $\mathbf{Q}$ to have only N non-zero entries; we term denote such matrices as N-sparse. Note that if we set, $N = 1$, we obtain the gridding reconstruction. The computational complexity of the reconstruction algorithm is dependent on N; it is desirable to minimize the approximation error $\|\mathbf{P}\mathbf{Q} - \mathbf{I}\|_F^2$, while keeping N as low as possible.

We decouple the error criterion in (8) as

$$\|\mathbf{Q}\mathbf{P} - \mathbf{I}\|_F^2 = \|\underbrace{\mathbf{P}^T}_{\mathbf{P}}\mathbf{Q}^T - \mathbf{I}\|_F^2 = \sum_{i=1}^{M}\|\mathbf{P}\mathbf{q}_i - \delta_i\|^2, \quad (9)$$

where $\mathbf{q}_i$ is the i^{th} row of $\mathbf{Q}$ and δ is the Kroneker delta function. Since each term of the summation is independent, we can solve for $\mathbf{q}_i, i = 0, .., N-1$ independently.

C. Determination of the optimal sparsity pattern

Classical schemes such as BURS and kSPA choose the non-zero entries of a row to be the N non-uniform samples that are nearest to the sample (nearest neighbor selection). As discussed previously, the computational complexity of the algorithm is dependent on how many neighbors are involved in the weighted summation. We aim to improve the accuracy for a specified computational complexity (i.e, N fixed) by determining the optimal sparsity pattern (see Fig 1). We propose to use the orthogonal matching pursuits (OMP) algorithm, which is a canonical greedy algorithm used for signal recovery from sparse measurement matrices [10].

The OMP algorithm to determine $\mathbf{q}_i$ with the optimal sparse support (location of non-zero entries, indicated by T) proceeds sequentially, increasing the support of $\mathbf{q}_i$ by one at each step, starting with the initialization $\mathbf{q}_i = 0; T = \{\}$. At each step, one performs the following operations to increment the support by one.

1) Determine the proxy as

$$\mathbf{y} = \mathbf{P}^T\underbrace{(\mathbf{P}\mathbf{q}_i - \delta_i)}_{\mathbf{r}_i}.$$

The magnitude of the proxy at different samples is indicative of the potential decrease in approximation error, if that term is made non-zero. For large images, the computation and storage of $\mathbf{P}$ is not possible, even on modern computers. Hence, we propose to compute the proxy using "on the fly computations":

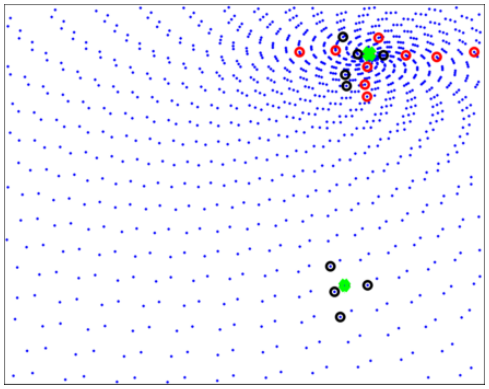
$$\mathbf{P}\mathbf{q}_i = \text{NUFFT}\left(\text{INUFFT}\left(\mathbf{q}_i\right)\right) + \lambda\mathbf{q}_i \quad (10)$$

and $\mathbf{P}^T\mathbf{r}$ as

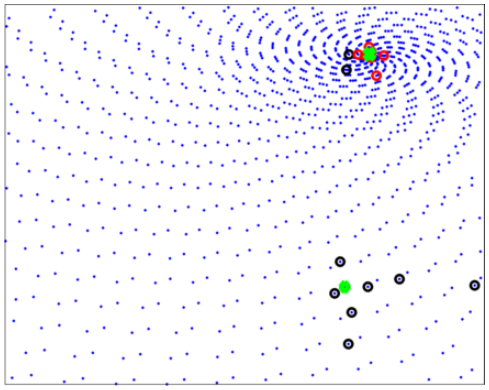
$$\mathbf{P}^T\mathbf{r} = \text{NUFFT}\left(\text{INUFFT}\left(\mathbf{r}\right)\right) + \lambda\mathbf{r} \quad (11)$$

The NUFFT operation involves the computation of the FFT of a sequence followed by interpolating the FFT onto a non-uniform grid [7]. This approch makes the algorithm computationally feasible for large image sizes.

2) The location corresponding to the maximum value of $\mathbf{y}$ is used to update the support T. i.e, we will update $T = \{T, \arg\max_i |\mathbf{y}_i|\}$.



(a) Unregularized sparsity pattern



(b) Regularized sparsity pattern

Fig. 1. We focus on two non-uniform Fourier sampling locations indicated by green dots in the above figures. The sparsity pattern corresponding to the dot in the center is indicated in red, while that of the one of the periphery is shown in black. Note that the sparsity patterns does not correspond to the immediate neighbors as assumed by [kspa,burs]. It is interesting to note that the sparsity pattern corresponding to the sample on the periphery also includes samples in the center of k-space. The added degree of freedom significantly decreases the reconstruction errors as shown Fig. 2

3) Determine the optimal $\mathbf{q}_i$, assuming the support to be T. This is obtained as

$$\mathbf{q}_i = \arg\min_{\mathbf{q}_i} \|\mathbf{P}_T \mathbf{q}_i - \delta_i\|^2 \tag{12}$$

Here, $\mathbf{P}_T$ is a sub-matrix of $\mathbf{P}$ obtained by picking the rows of $\mathbf{P}$ corresponding to the support T. Again, since it is often difficult to compute and store $\mathbf{P}$, we compute the i^{th} row of $\mathbf{P}_T$ as

$$\mathbf{P}_{T_i} = \mathbf{P}\delta_{T_i} = \text{NUFFT}\left(\text{INUFFT}\right)\left(\delta_{T_i}\right) + \lambda\delta_{T_i} \tag{13}$$

The i^{th} row of $\mathbf{P}_T$ is determined by setting the T_i^{th} non-uniform sample by one and then computing the INUFFT and NUFFT as described above. We determine $\mathbf{q}_i$ as $\mathbf{q}_i = \mathbf{P}_T^{\#}\delta_i$, where $\mathbf{P}^{\#}$ indicates the pseudo-inverse of $\mathbf{P}$.

The above operations are repeated until a specified support size is achieved. It was also observed that beyond a certain support size (see Fig.2), the reduction in the reconstruction error was negligible in comparison to the increase in the computation time. If we set the support size to be one, we will obtain a scheme similar to gridding; the weights will correspond to the optimal density compensation functions.

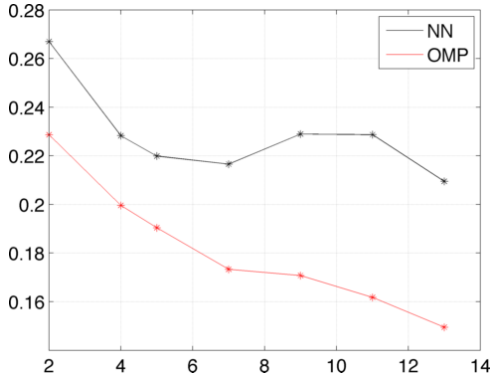


Fig. 2. Error with increasing support sizes. The error due to the supports chosen using nearest neighbor is not monotonically decreasing. Since it is a heuristic approach to selecting supports it may not work well with all types of sampling trajectories.

III. Results

In this section, we will analyze the proposed scheme. Specifically, we will study the sparsity patterns and the improvement obtained by optimizing the patterns over selecting nearest neighbors. We will also compare the proposed schemes conjugate gradient and gridding reconstructions. For all the studies, we rely on the two dimensional 64x64 Shepp-Logan phantom and assumed a spiral trajectory.

A. Sparsity patterns of the $\mathbf{Q}$ matrix

We study the effect of sparsity patterns where we focus on regularized and unregularized reconstructions. We focus on two regions, one at the centre of *k* space and the other at the periphery. For optimal supports, points located at the periphery have supports at the centre of k space (Fig.1). The standard nearest neighbor selection fails to capture these points and thus results in higher reconstruction errors as shown in Fig 2. Also for the regularized case we see that the sparsity pattern in the central regions is more clustered than for the unregularized case.

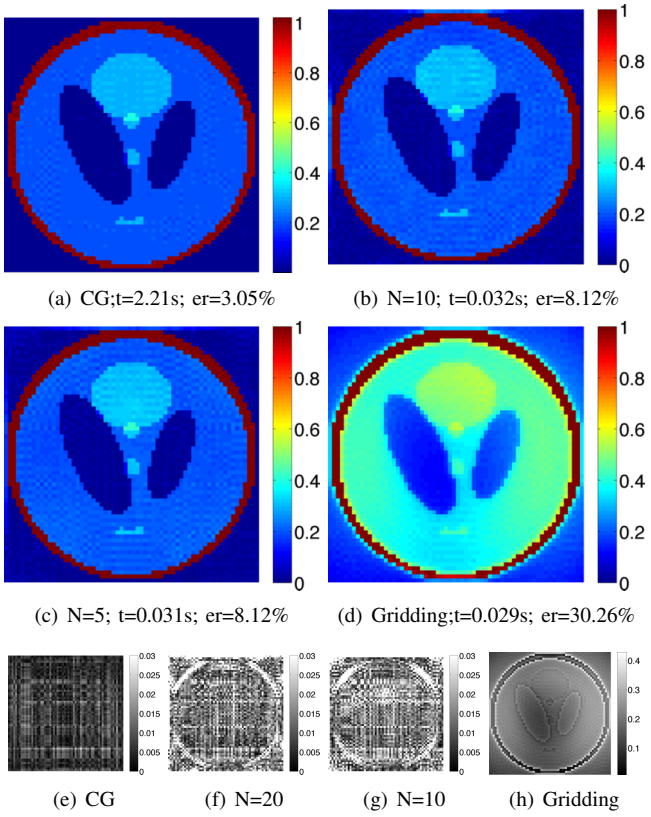
B. Unregularized reconstruction

We will now compare the performance of the algorithm in a Nyquist sampled trajectory with $\lambda = 0$. We performed the reconstructions using standard CG algorithm, the proposed scheme at N=10 and N=5, as well as the gridding. The results are shown in Fig. 3. Note that the CG reconstructions are well approximated by the proposed scheme, while the computation time of the proposed scheme is lower by a factor of 70. The computation time of the proposed scheme is close to that of gridding. However, the reconstruction errors are significantly reduced over gridding.

C. Regularized reconstruction

To study the utility of regularization, we consider a four-fold under-sampled trajectory. The comparisons are shown in Fig. 4. Note that the CG reconstruction and the proposed

scheme with $N = 20$ are in good agreement. Both the algorithms are capable of recovering the small features, inspite of the four-fold under-sampling. In contrast, the gridding reconstructions have significant aliasing artifacts. The ability to perform regularized reconstructions in a computation time that is comparable to gridding is a significant benefit in dynamic MRI applications, where acquiring fully sampled data might result in lower temporal resolution.



(a) CG;t=2.21s; er=3.05% (b) N=10; t=0.032s; er=8.12%

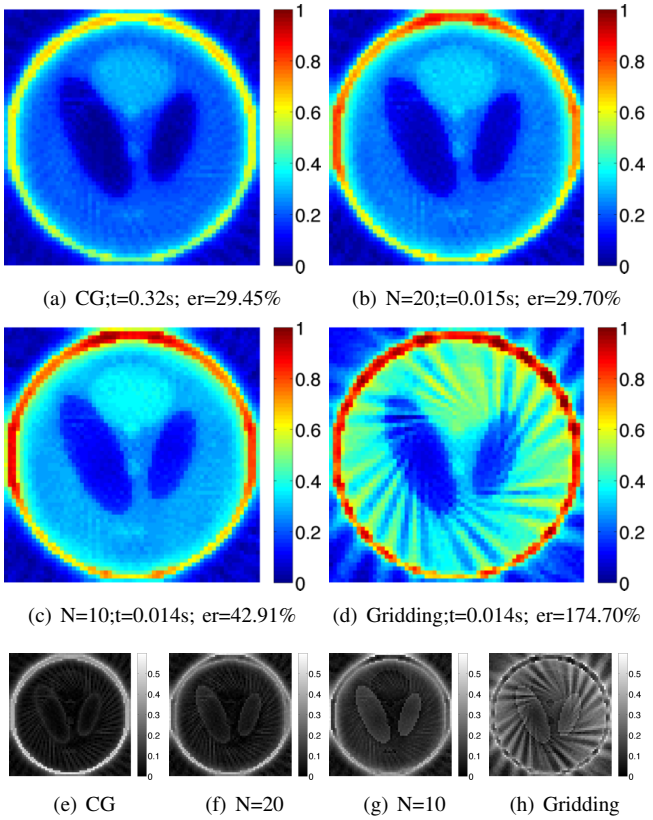
(c) N=5; t=0.031s; er=8.12% (d) Gridding;t=0.029s; er=30.26%

(e) CG (f) N=20 (g) N=10 (h) Gridding

Fig. 3. Comparison of reconstructions from Nyquist sampled non-Cartesian data, obtained using different algorithms. The proposed reconstructions are comparable to that obtained using CG.

IV. Conclusions

We proposed a novel algorithm to reconstruct non Cartesian MRI data. We reformulated the recovery as the multiplication by a sparse matrix, followed by a non-uniform IFFT. We derived the optimal sparsity pattern of the matrix, thus decreasing the reconstruction error for a specified computational complexity. The reconstructions obtained by the proposed algorithm is a good approximation to the one obtained by using conjugate gradients algorithm. However, the computational complexity of the proposed scheme is comparable to gridding schemes, which is significantly lower than conjugate gradients methods. The ability to account for regularization makes the proposed approach ideal for the reconstruction of large undersampled, noisy dynamic/functional MRI datasets.



(a) CG;t=0.32s; er=29.45% (b) N=20;t=0.015s; er=29.70%

(c) N=10;t=0.014s; er=42.91% (d) Gridding;t=0.014s; er=174.70%

(e) CG (f) N=20 (g) N=10 (h) Gridding

Fig. 4. Comparison of regularized reconstruction schemes. Here, we subsampled the trajectory used for the previous experiment by four fold. The direct use of gridding algorithm results in significant aliasing. In contrast, CG and the proposed schemes can reasonably recover the smaller details. The computational complexity of the proposed scheme is comparable to gridding. Due to the decreased number of samples, fewer Cartesian to non-Cartesian interpolations are required; the computation time of CG is lower than the fully sampled case.

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