Ch9. Flow over Immersed Bodies (2)

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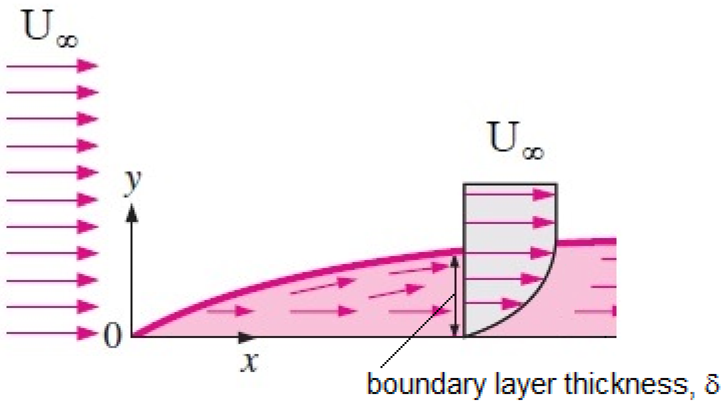
Review of Flat Plate BL Theory



BL equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

B.C.'s:

$$u = v = 0 \quad @y = 0$$
$$u = U_\infty \quad @y = \delta$$

Blasius solution for laminar BL:

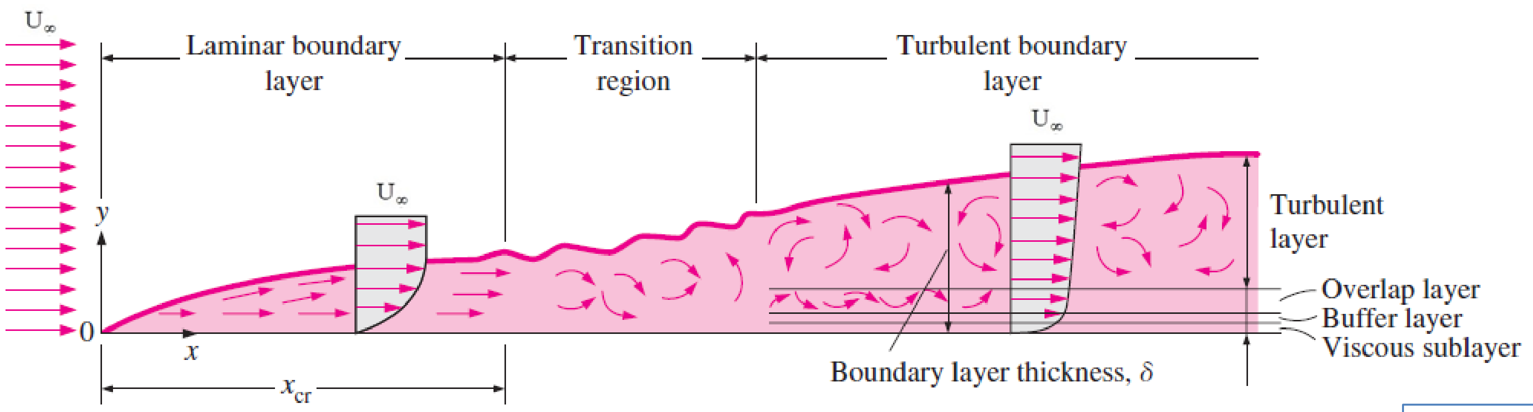
$$\delta = \frac{5x}{\sqrt{\mathrm{Re}_x}}$$

$$c_f(x) = \frac{2\tau_w}{\rho U_\infty^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_f = \frac{2D_f}{\rho U_\infty^2 bL} = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

$$\therefore D_f = C_f \cdot \frac{1}{2}\rho U_\infty^2 bL$$

Transition to Turbulent BL



$Re_x = 10^5 \sim 3\times10^6 \Rightarrow Re_{cr} = 5\times10^5$

$$Re_{cr} = \frac{U_\infty x_{cr}}{\nu}$$

- Transition from laminar to turbulent flow begins at about $Re_x \approx 10^5$, but does not become fully turbulent before $Re_x \approx 3 \times 10^6$. In engineering analysis, a generally accepted value for the critical Reynolds number is $\mathbf{Re_{cr} = 5 \times 10^5}$ (also referred to as Re_{tr}).
- The velocity profile in turbulent flow is much fuller than in laminar flow, with a sharp drop near the surface:
  - **Viscous sublayer**: Viscous effects are dominant
  - **Buffer layer**: Turbulent effects become significant but viscous effects still dominant
  - **Overlap layer**: Turbulent effects are much more significant, but still not dominant
  - **Turbulent layer**: Turbulent effects dominate over viscous effects

Example

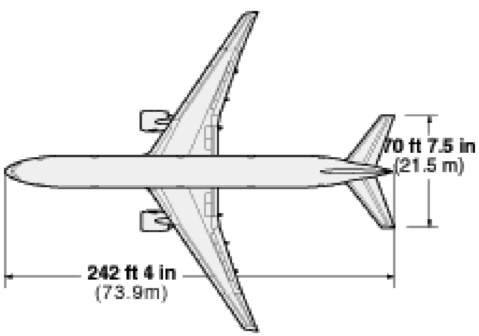
9.2 6 An airplane flies at a speed of 400 mph at an altitude of 10,000 ft. If the boundary layers on the wing surfaces behave as those on a flat plate, estimate the extent of laminar boundary layer flow along the wing. Assume a transitional Reynolds number of $\mathrm{Re}_{xcr} = 5 \times 10^5$.

$$\mathrm{Re}_{\mathrm{cr}} = \frac{U_\infty x}{\nu} = 5 \times 10^5$$

$$U_\infty = (400\ \mathrm{mph})\left(\frac{1\ \mathrm{hr}}{3600\ \mathrm{s}}\right)\left(\frac{5280\ \mathrm{ft}}{\mathrm{mi}}\right) = 587\ \ \mathrm{ft/s}$$

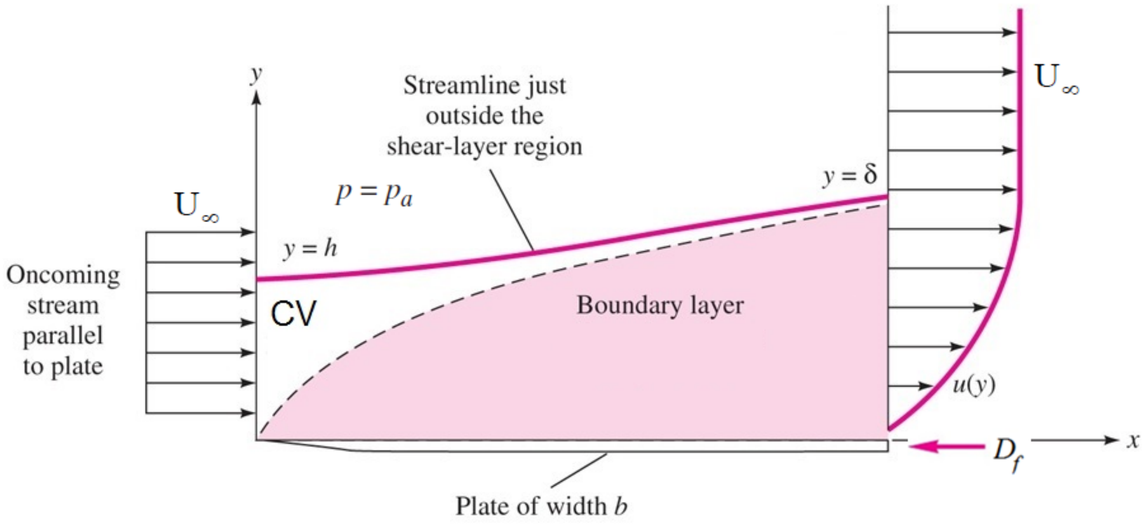
$$\nu = 2.01 \times 10^{-4}\ \ \mathrm{ft^2/s}$$

$$\therefore x_{\mathrm{cr}} = \frac{\nu \cdot \mathrm{Re}_{\mathrm{cr}}}{U_\infty} = \frac{(2.01 \times 10^{-4})(5 \times 10^5)}{(587)} = 0.171\ \mathrm{ft}$$



Boeing 777 (www.aerospaceweb.org)

Momentum Integral Analysis



Momentum equation:

$$(\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}} = \sum F_x$$

or

$$\int_0^{\delta} u \cdot \rho u b dy - (\rho U_\infty b h) \cdot U_\infty = -D_f \qquad -(1)$$

Momentum Integral Analysis – Contd.

Continuity equation:

$$\dot{m}_{\text{out}} - \dot{m}_{\text{in}} = 0$$

or

$$\int_0^{\delta} \rho u b dy - \rho U_{\infty} b h = 0$$

Solve for h,

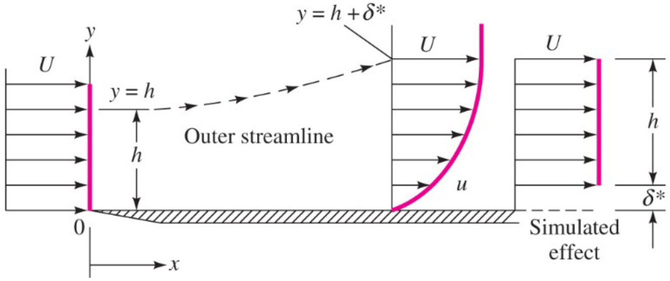
$$\therefore h = \int_0^{\delta} \frac{u}{U_{\infty}} dy \qquad -(2)$$

<u>Note: Displacement thickness</u>

$$\delta^* = \delta - h = \int_0^{\delta} dy - \int_0^{\delta} \frac{u}{U_{\infty}} dy$$

$$\therefore \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U_{\infty}}\right) dy$$

δ^* is a measure of displacement of inviscid flow due to BL.



Momentum Integral Analysis – Contd.

Substitute (2) into (1), then

$$D_f = \rho b \int_0^{\delta} u(U_\infty - u)dy \quad -(3)$$

Kármán (1921) wrote (3) in a convenient form by using momentum thickness θ:

$$D_f = \rho U_\infty^2 b\theta \quad -(4)$$

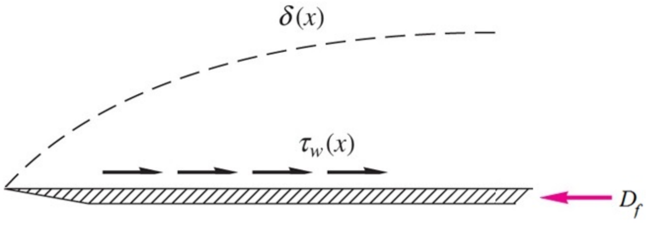
where,

$$\theta \equiv \int_0^{\delta} \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy$$

Momentum Integral Analysis – Contd.

Kármán then noted that the drag also equals the integrated wall shear stress along the plate:

$$D_f(x) = \int_0^x \tau_w(x) b\,dx$$



or

$$\frac{dD_f}{dx} = \tau_w b \qquad -(5)$$

$$c_f = \frac{2\tau_w}{\rho U_\infty^2}$$

By comparing (5) with the derivative of (4)

$$\tau_w = \rho U_\infty^2 \frac{d\theta}{dx}$$

or, in a non-dimensional form,

$$c_f = 2\frac{d\theta}{dx} \qquad -(6)$$

This is called the **momentum integral relation** for flat-plate boundary layer flow.
It is valid for either laminar or turbulent flat-plate flow.

Approximate Solution for Laminar BL

Simple velocity-profile approximation:

$$u = U_\infty \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$$

Then,

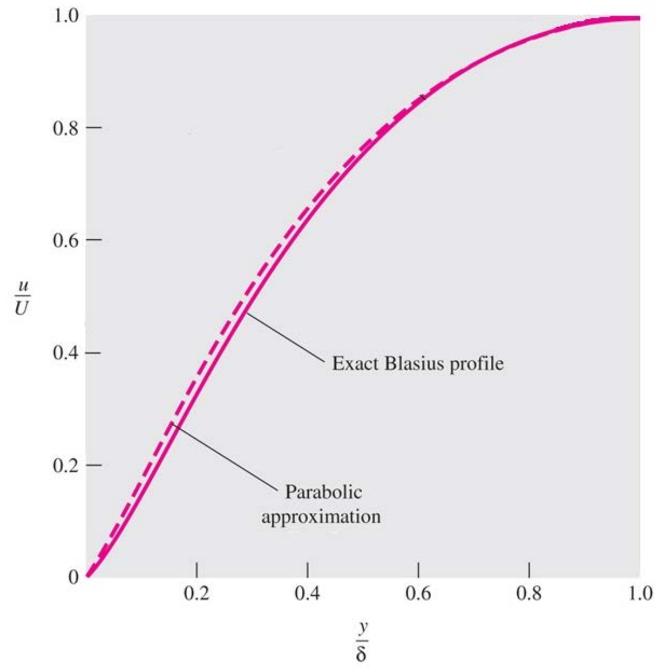
$$\theta = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{2y}{\delta} + \frac{y^2}{\delta^2}\right) = \frac{2}{15}\delta \quad -(7)$$

and

$$\tau_w = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \frac{2\mu U_\infty}{\delta}$$

or

$$c_f = \frac{4\nu}{U_\infty \delta} \quad -(8)$$



Substitute (7) and (8) into the momentum integral relation (6),

$$\frac{4\nu}{U_\infty \delta} = 2\frac{d}{dx}\left(\frac{2}{15}\delta\right) \quad -(9)$$

Approximate Solution for Laminar BL

By solving the differential equation (9) for δ,

$$\delta = \frac{5.5x}{\sqrt{\mathrm{Re}_x}}$$

Also, from equation (8),

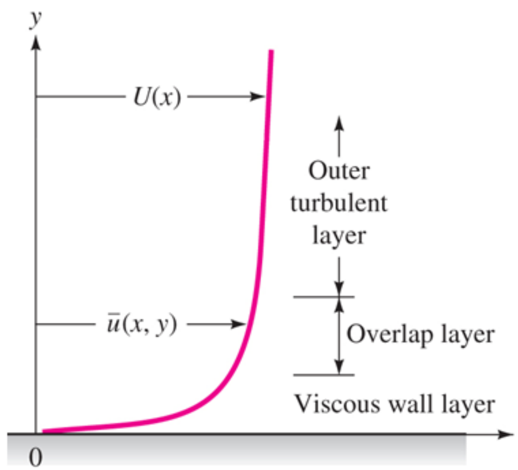
$$c_f = \frac{0.73}{\sqrt{\mathrm{Re}_x}}$$

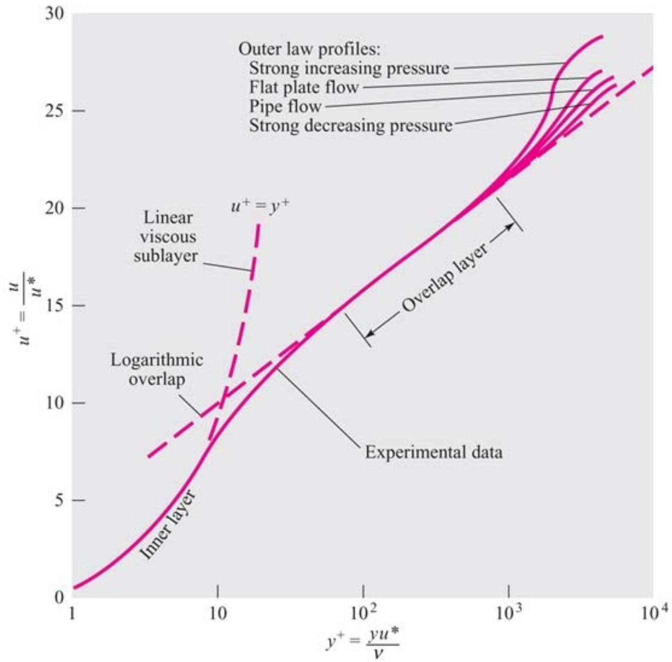
and

$$C_f = \frac{1.46}{\sqrt{\mathrm{Re}_L}}$$

About 10% error to the Blasius solution

Turbulent BL





Turbulent flat-plate flow is nearly logarithmic, with a slight outer wake and a think viscous sublayer,

$$\frac{u}{u^*} = \frac{1}{\kappa} \ln \frac{yu^*}{\nu} + B \qquad u^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

with κ = 0.41 and B = 5.0. Therefore, we assume that the logarithmic law holds all the way across the boundary layer

Turbulent BL – Contd.

At the outer edge of the BL, $y = \delta$ and $u = U_\infty$, the logarithmic law becomes,

$$\frac{U_\infty}{u^*} = \frac{1}{\kappa} \ln \frac{\delta u^*}{\nu} + B \qquad -(10)$$

Here,

$$\frac{U_\infty}{u^*} = \frac{U_\infty}{\sqrt{\tau_w/\rho}} = \left(\frac{2}{2\tau_w/\rho U_\infty^2}\right)^{\frac{1}{2}} = \left(\frac{2}{c_f}\right)^{\frac{1}{2}}$$

and

$$\frac{\delta u^*}{\nu} = \frac{\delta\sqrt{\tau_w/\rho}}{\nu} = \frac{U_\infty \delta}{\nu}\sqrt{\frac{2\tau_w}{\rho U_\infty^2}} = \mathrm{Re}_\delta \left(\frac{c_f}{2}\right)^{\frac{1}{2}}, \qquad \mathrm{Re}_\delta \equiv \frac{U_\infty \delta}{\nu}$$

Then, (10) is a skin friction law for turbulent flat-plate flow:

$$\left(\frac{2}{c_f}\right)^{\frac{1}{2}} = 2.44 \ln\left[\mathrm{Re}_\delta \left(\frac{c_f}{2}\right)^{\frac{1}{2}}\right] + 5.0$$

Approximate Solutions for Turbulent BL (1)

Prandtl's one-seventh-power law:

$$\frac{u}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7}$$

Then,

$$\theta = \int_0^\delta \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left(1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right) dy = \frac{7}{72}\delta \quad - (11)$$

power-law approximation:

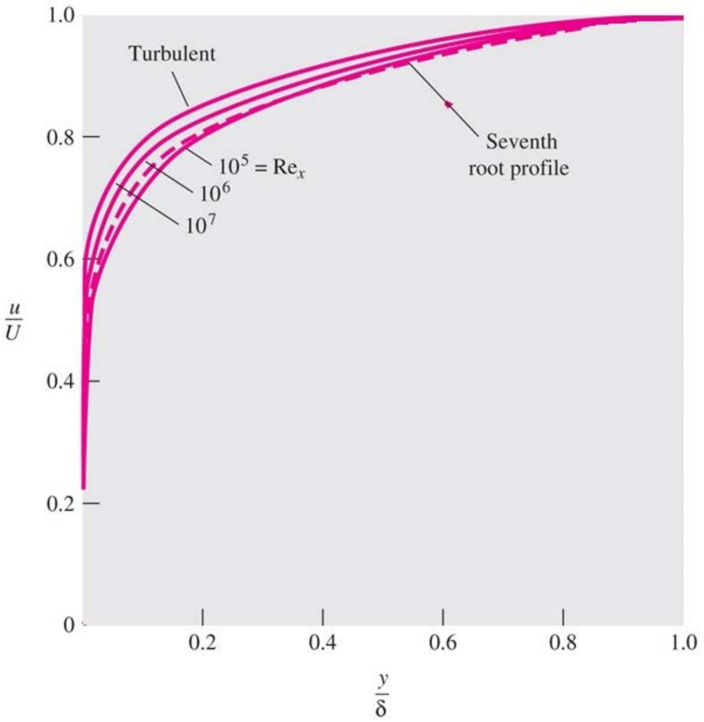
$$c_f \approx 0.02\,\mathrm{Re}_\delta^{-\frac{1}{6}} \qquad - (12)$$

Substitute (12) and (11) into Kaman's momentum law (6),

$$0.02\mathrm{Re}_\delta^{-1/6} = 2\frac{d}{dx}\left(\frac{7}{72}\delta\right) \quad - (13)$$

By solving the differential equation (13) for δ,

$$\delta = \frac{0.16x}{\mathrm{Re}_x^{1/7}} \qquad - (14)$$



Approximate Solutions for Turbulent BL (1) – Contd.

By combining (14) with the power-law approximation (12),

$$c_f = \frac{0.027}{\mathrm{Re}_x^{1/7}}$$

Friction drag coefficient,

$$C_f = \frac{2D_f}{\rho U_\infty^2 bL} = \frac{1}{L}\int_0^L c_f\, dx$$

$$\therefore C_f = \frac{0.031}{\mathrm{Re}_L^{1/7}}$$

Note: These formulas are for a fully turbulent flow over a smooth flat plate from the leading edge; in general, give better results for sufficiently large Reynold number $\mathrm{Re}_L > 10^7$.

Example

9.81 If the wetted area of an 80-m ship is 1500 m^2, approximately how great is the surface drag when the ship is traveling at a speed of 10 m/s? What is the thickness of the boundary layer at the stern? Assume $T = 10°C$.

$$\mathrm{Re}_L = \frac{U_\infty L}{\nu} = \frac{(10)(80)}{1.4 \times 10^{-6}} = 5.7 \times 10^8$$

$$C_f = \frac{0.031}{\mathrm{Re}_L^{1/7}} = \frac{0.031}{(5.7 \times 10^8)^{1/7}} = 0.00174$$

$$\therefore D_f = C_f \cdot \frac{1}{2}\rho U_\infty^2 A = (0.00174)\left(\frac{1}{2}\right)(1{,}026)(10)^2(1{,}500) = 134\ \mathrm{kN}$$

$$\delta(L) = \frac{0.16L}{\mathrm{Re}_L^{1/7}} = \frac{(0.16)(80)}{(5.7 \times 10^8)^{1/7}} = 0.718\ \mathrm{m}$$

Approximate Solutions for Turbulent BL (2)

Alternate forms are by using the same velocity profile $u/U_\infty = (y/\delta)^{1/7}$ assumption but using an experimentally determined shear stress formula $\tau_w = 0.0225\rho U_\infty^2 (\nu/U_\infty \delta)^{1/4}$,

$$\frac{\delta}{x} = \frac{0.37}{\mathrm{Re}_x^{1/5}}$$

$$c_f = \frac{0.058}{\mathrm{Re}_x^{1/5}}$$

$$C_f = \frac{0.074}{\mathrm{Re}_x^{1/5}}$$

These formulas are valid only in the range of the experimental data, which covers $\mathrm{Re}_L = 5 \times 10^5 \sim 10^7$ for smooth flat plates.

Approximate Solutions for Turbulent BL (3)

Other empirical formulas are by using the logarithmic velocity profile instead of the 1/7-power law:

$$\frac{\delta}{L} = c_f(0.98 \log \mathrm{Re}_L - 0.732)$$

$$c_f = (2 \log \mathrm{Re}_x - 0.65)^{-2.3}$$

$$C_f = \frac{0.455}{(\log_{10} \mathrm{Re}_L)^{2.58}}$$

These formulas are also called as the Prandtl-Schilichting skin-friction formula and valid in the whole range of $\mathrm{Re}_L \leq 10^9$.

Composite Formulas

To take into account both the initial laminar boundary layer and subsequent turbulent boundary layer, i.e., in the transition region ($5 \times 10^5 < \mathrm{Re}_L < 8 \times 10^7$) where the laminar drag at the leading edge is an appreciable fraction of the total drag:

$$C_f = \frac{0.031}{\mathrm{Re}_L^{1/7}} - \frac{1440}{\mathrm{Re}_L}$$

or

$$C_f = \frac{0.074}{\mathrm{Re}_L^{1/5}} - \frac{1700}{\mathrm{Re}_L}$$

or

$$C_f = \frac{0.455}{(\log_{10} \mathrm{Re}_L)^{2.58}} - \frac{1700}{\mathrm{Re}_L}$$

with transitions at $\mathrm{Re}_{\mathrm{tr}} = 5 \times 10^5$ for all cases.

Friction Drag Coefficient for Flat-plate BL