Fluids in Rigid-Body Motion

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Newton's 2nd Law of Motion

- In general, for a body of mass m,

$$m\boldsymbol{a} = \sum \boldsymbol{F}$$

where, $\boldsymbol{a}$ is the acceleration of the body and $\sum \boldsymbol{F}$ is the vector sum of the external forces acting on the body.

- For a fluid element,

$$m\boldsymbol{a} = \boldsymbol{F}_{\mathrm{B}} + \boldsymbol{F}_{\mathrm{S}} \qquad (1)$$

where,
  - $\boldsymbol{F}_{\mathrm{B}}$ is the body force due to the gravity, i.e., the weight of the fluid element
  - $\boldsymbol{F}_{\mathrm{S}}$ is the surface force due to the pressure and viscous friction on the surface of the fluid element

- In fluids, often times the motion equation is written for a unit volume by using the relationship $m = \rho V\!\!\!-$ and dividing Eq. (1) by the volume $V\!\!\!-$,

$$\rho\boldsymbol{a} = \boldsymbol{f}_{\mathrm{b}} + \boldsymbol{f}_{\mathrm{s}}$$

where, $\boldsymbol{f}_{\mathrm{b}}$ and $\boldsymbol{f}_{\mathrm{s}}$ are the body and surface forces per unit volume.

Newton's 2nd Law of Motion – Contd.

- Body force (Weight of the fluid)

$$\boldsymbol{F}_B = -W\hat{\boldsymbol{k}} = -\rho g V\hat{\boldsymbol{k}}$$

$$\therefore \boldsymbol{f}_b = -\rho g\hat{\boldsymbol{k}}$$

- Surface force

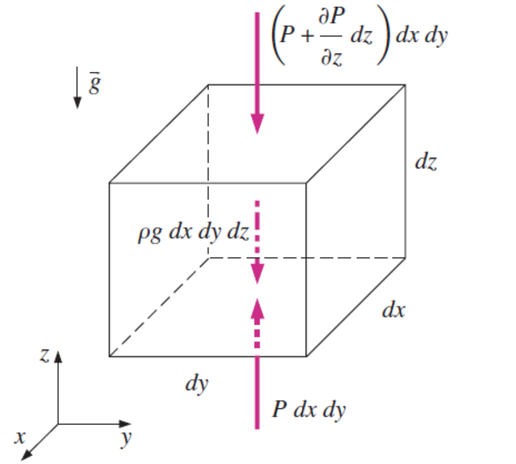
$$\boldsymbol{f}_s = \boldsymbol{f}_p + \boldsymbol{f}_v$$

where,
  - $\boldsymbol{f}_p = -\nabla p$ due to the pressure
  - $\boldsymbol{f}_v = \nabla \cdot \boldsymbol{\tau}$ due to the viscous shear stress

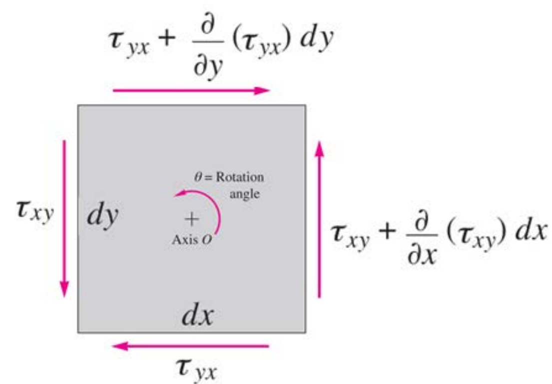
- General motion equation for fluids

$$\rho \boldsymbol{a} = -\rho g\hat{\boldsymbol{k}} - \nabla p + \nabla \cdot \boldsymbol{\tau} \qquad (2)$$

Note: For one dimensional flow of Newtonian fluids, $\tau = \mu \frac{du}{dy}$. This implies that the viscous **shear stress** (or the shear force) is caused by **the relative motion between fluid particles**.



The body force and the surface pressure force acting on a differential fluid element in the vertical direction.



Shear stresses that may cause a net angular acceleration about axis *O*.

Special Case: Fluids at Rest

- For fluids at rest, i.e., with no motion, Eq. (2) can be simplified as

$$\underbrace{\rho \boldsymbol{a}}_{=0} = -\rho g \widehat{\boldsymbol{k}} - \nabla p + \underbrace{\nabla \cdot \boldsymbol{\tau}}_{=0}$$

or,

$$\nabla p = \rho \mathbf{g} \quad (3)$$

where, $\mathbf{g} = -g\hat{\mathbf{k}}$.

- If rewrite Eq. (3) in components,

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g \quad (4)$$

Thus, p is independent of x and y (i.e., the pressure remains constant in any horizontal direction) and varies only in the vertical direction z as a result of gravity.

- If ρ is constant, the solution of Eq. (4) becomes

$$p = -\gamma z$$

by taking $p = 0$ at $z = 0$. This is the hydrostatic pressure equation for incompressible fluids at rest.

Rigid Body Motion

- In rigid-body motion, all particles are in combined translation and rotation, and there is no relative motion between particles.
- With no relative motion, there are no strains or strain rates, so that the viscous term in Eq. (2) vanishes,

$$\rho \boldsymbol{a} = -\rho \mathrm{g} \widehat{\boldsymbol{k}} - \nabla p + \underbrace{\nabla \cdot \boldsymbol{\tau}}_{=0}$$

or,

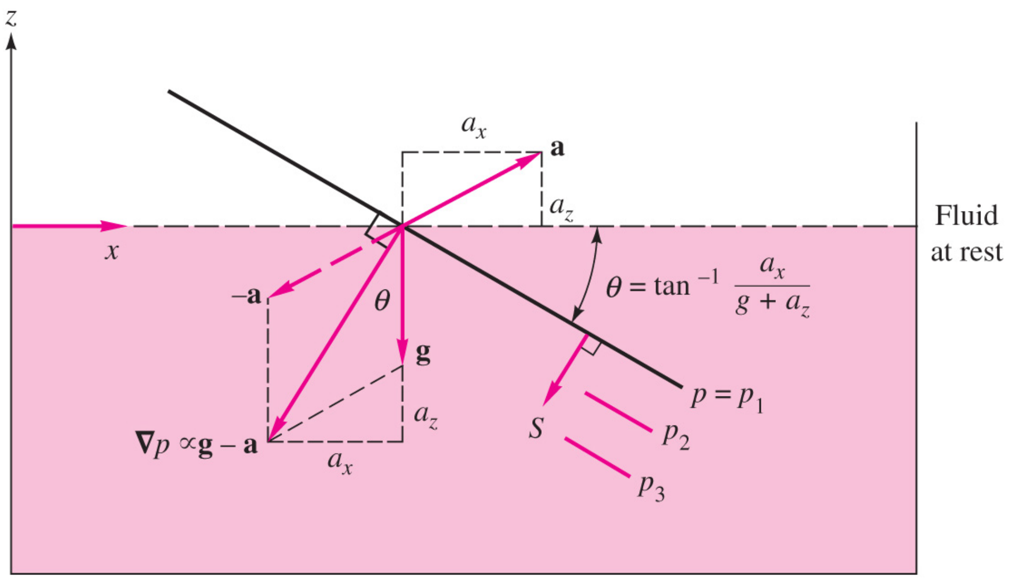
$$\nabla p = \rho(\mathbf{g} - \boldsymbol{a}) \qquad (5)$$

where, $\mathbf{g} = -\mathrm{g}\widehat{\boldsymbol{k}}$.

- Two simple rigid-motion cases of interest are
  a) Rigid body translation: Constant linear acceleration $\boldsymbol{a} = a_x \hat{\boldsymbol{\imath}} + a_z \widehat{\boldsymbol{k}}$
  b) Rigid body rotation: Constant rotation $\boldsymbol{\Omega} = \Omega \widehat{\boldsymbol{k}}$

Rigid Body Translation

- In case of uniform rigid-body acceleration, Eq. (5) applies, $\boldsymbol{a}$ having the same magnitude and direction for all particles.
- The vector sum of $\mathbf{g}$ and $-\boldsymbol{a}$ gives the direction of the pressure gradient or the greatest rate of increase of p.
- Then, the surfaces of constant pressure must be perpendicular to the direction of pressure gradient and are thus tilted at a downward angle θ.



Tilting of constant-pressure surfaces in a tank of liquid in rigid-body acceleration.

$$\nabla p = \rho(\mathbf{g} - \boldsymbol{a}) \quad (5)$$

where,

$$\mathbf{g} = -\mathrm{g}\hat{\boldsymbol{k}}$$

$$\boldsymbol{a} = a_x \hat{\boldsymbol{\imath}} + a_z \hat{\boldsymbol{k}}$$

Thus,

$$\nabla p = \frac{\partial p}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{k}} = -\rho a_x \hat{\boldsymbol{\imath}} - \rho(\mathrm{g} + a_z)\hat{\boldsymbol{k}}$$

Equating like components,

$$\frac{\partial p}{\partial x} = -\rho a_x \qquad \frac{\partial p}{\partial z} = -\rho(\mathrm{g} + a_z)$$

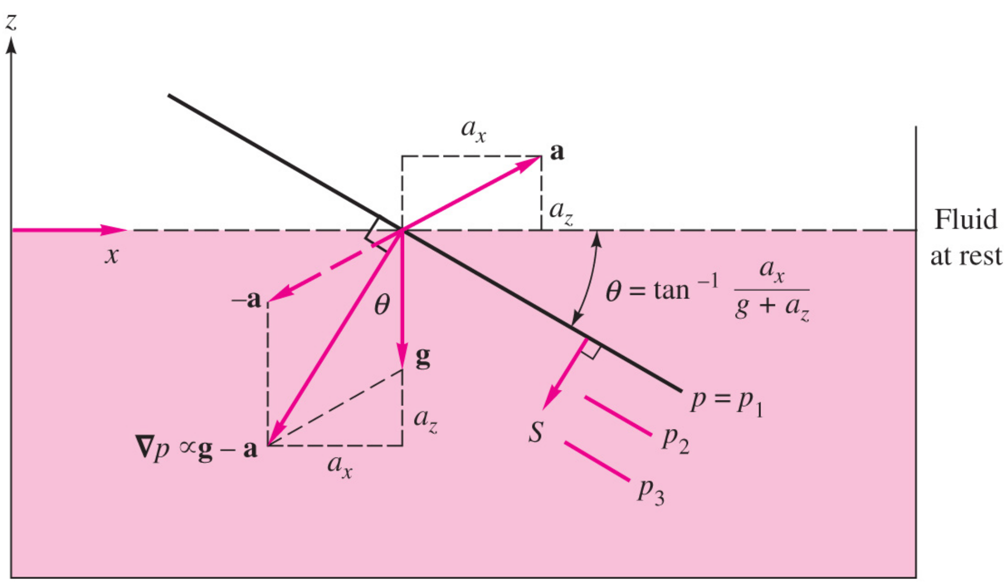
The angle of constant pressure lines,

$$\theta = \tan^{-1}\frac{a_x}{\mathrm{g} + a_z}$$

Rigid Body Translation – Contd.

- One of the tilted lines (the surfaces of constant pressure) is the free surface, which is found by the requirement that the fluid retain its volume unless it spills.
- The rate of increase of pressure in the direction $\mathbf{g} - \boldsymbol{a}$ is greater than in the ordinary hydrostatics and is given by

$$\frac{dp}{ds} = \rho G \quad \text{where } G = \sqrt{a_x^2 + (\mathrm{g} + a_z)^2}$$



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where,

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Thus,

$$\nabla p = \frac{\partial p}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{k}} = -\rho a_x \hat{\boldsymbol{\imath}} - \rho(\mathrm{g} + a_z)\hat{\boldsymbol{k}}$$

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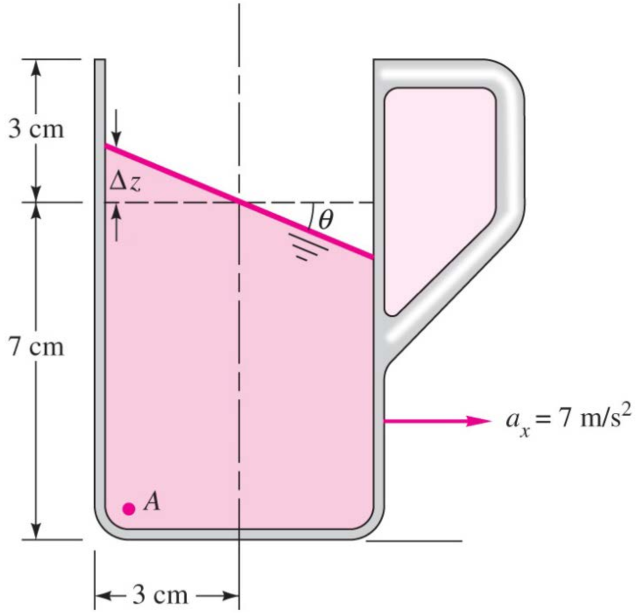
Rigid Body Translation – Example

EXAMPLE 2.13

A drag racer rests her coffee mug on a horizontal tray while she accelerates at 7 m/s^2. The mug is 10 cm deep and 6 cm in diameter and contains coffee 7 cm deep at rest. (*a*) Assuming rigid-body acceleration of the coffee, determine whether it will spill out of the mug. (*b*) Calculate the gage pressure in the corner at point *A* if the density of coffee is 1010 kg/m^3.



The coffee tilted during the acceleration.

$$\theta = \tan^{-1}\frac{a_x}{g} = \tan^{-1}\frac{7}{9.81} = 35.5°$$

$$\Delta z = (3)(\tan 35.5°) = 2.14 \text{ cm} < 3 \text{ cm} \;\; \text{(no spilling)}$$

$$p_A = \rho G \Delta s = (1010)\sqrt{(7)^2 + (9.81)^2}[(0.07 + 0.0214)\cos 35.5°] = 906 \text{ Pa}$$

(Note: When at rest, $p_A = \rho g h_{\text{rest}} = (1010)(9.81)(0.07) = 694$ Pa)

Alternatively, since $a_z = 0$ thus $\frac{\partial p}{\partial z} = -\rho g$,

$$p_A = \rho g \Delta z = (1010)(9.81)(0.07 + 0.0214) = 906 \text{ Pa}$$

Rigid Body Rotation

- For a fluid rotating about the z axis at a constant rate Ω without any translation, the fluid acceleration will be a centripetal term,

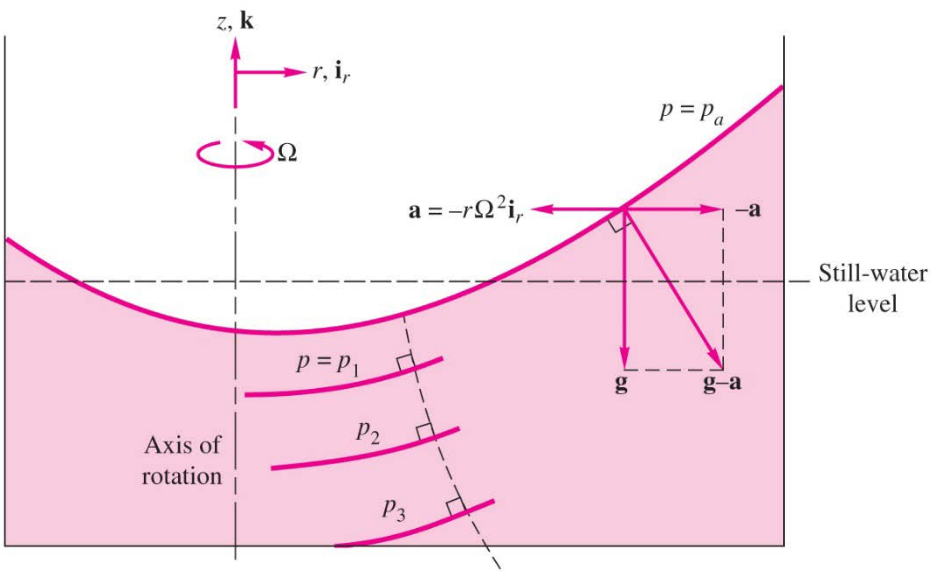
$$\boldsymbol{a} = -r\Omega^2 \hat{\boldsymbol{\imath}}_{\boldsymbol{r}}$$

- From Equation (5) written in a cylindrical coordinate system,

$$\nabla p = \frac{\partial p}{\partial r}\hat{\boldsymbol{\imath}}_{\boldsymbol{r}} + \frac{\partial p}{\partial z}\hat{\boldsymbol{k}} = \rho(\mathbf{g} - \boldsymbol{a}) = \rho\left(r\Omega^2\hat{\boldsymbol{\imath}}_{\boldsymbol{r}} - \mathrm{g}\hat{\boldsymbol{k}}\right)$$

- Equating like components,

$$\frac{\partial p}{\partial r} = \rho r \Omega^2 \qquad \frac{\partial p}{\partial z} = -\rho \mathrm{g} \qquad (6)$$



Development of paraboloid constant-pressure surfaces in a fluid in rigid-body rotation. The dashed line along the direction of maximum pressure increase is an exponential curve.

- By solving the two 1st-order PDE's in Eq. (6),

$$p = p_0 - \rho \mathrm{g} z + \frac{1}{2}\rho r^2 \Omega^2 \quad (7)$$

where, p_0 is the pressure at $(r, z) = (0,0)$.

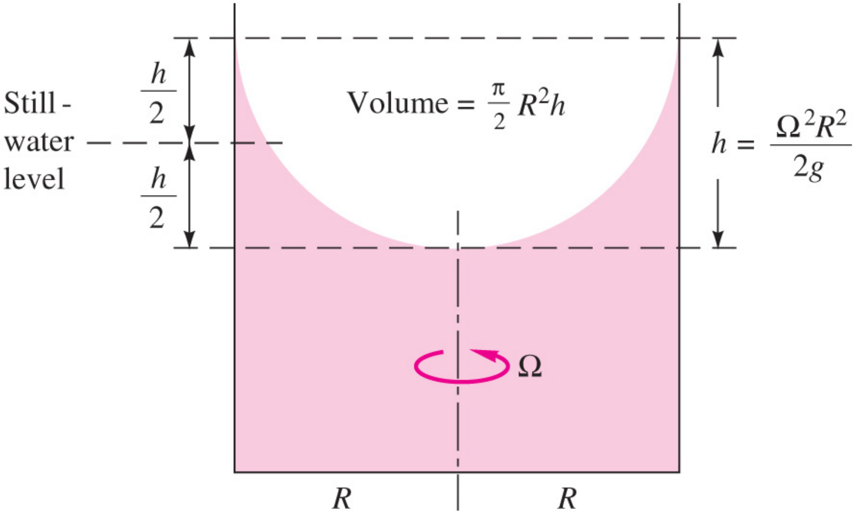
- The pressure is linear in z and quadratic (parabolic) in r.

Rigid Body Rotation – Contd.

- If we wish to plot a constant-pressure surface, say $p = p_1$, Equation (7) becomes

$$z = \frac{p_0 - p_1}{\rho g} + \frac{r^2\Omega^2}{2g} = a + br^2$$

- Thus, the surfaces are paraboloids of revolution, concave upward, with their minimum points on the axis of rotation.



Determining the free surface position for rotation of a cylinder of fluid about its central axis.

- Similarly as in rigid body translation case, the position of the free surface is found by conserving the volume of fluid.
- Since the volume of a paraboloid is one-half of the base area times its height, the still-water level is exactly halfway between the high and low points of the free surface.
- The center of the fluid drops an amount

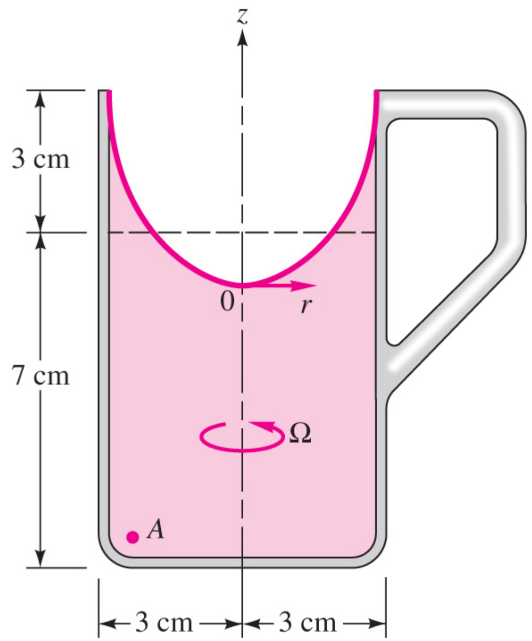
$$\frac{h}{2} = \frac{\Omega^2 R^2}{4g}$$

and the edges rise an equal amount.

Rigid Body Rotation – Example

EXAMPLE 2.14

The coffee cup in Example 2.13 is removed from the drag racer, placed on a turntable, and rotated about its central axis until a rigid-body mode occurs. Find (*a*) the angular velocity that will cause the coffee to just reach the lip of the cup and (*b*) the gage pressure at point *A* for this condition.



The coffee cup placed on a turntable.

$$\frac{h}{2} = \frac{\Omega^2 R^2}{4g} = \frac{\Omega^2 (0.03)^2}{4(9.81)} = 0.03$$

$$\therefore \Omega = 36.2\,\text{rad/s} = 345\text{ rpm}$$

Since point A is at $(r, z) = (3\text{ cm}, -4\text{ cm})$ and by putting the origin of coordinates r and z at the bottom of the free-surface depression, thus $p_0 = 0$ (i.e., gage pressure),

$$p_A = p_0 - \rho g z + \frac{1}{2}\rho r^2 \Omega^2$$
$$= 0 - (1010)(9.81)(-0.04) + \frac{1}{2}(1010)(0.03)^2(36.2)^2 = 990\text{ Pa}$$

(Note: This is about 43% greater than the still-water pressure p_A = 694 Pa)