Reynolds Transport Theorem and Continuity Equation

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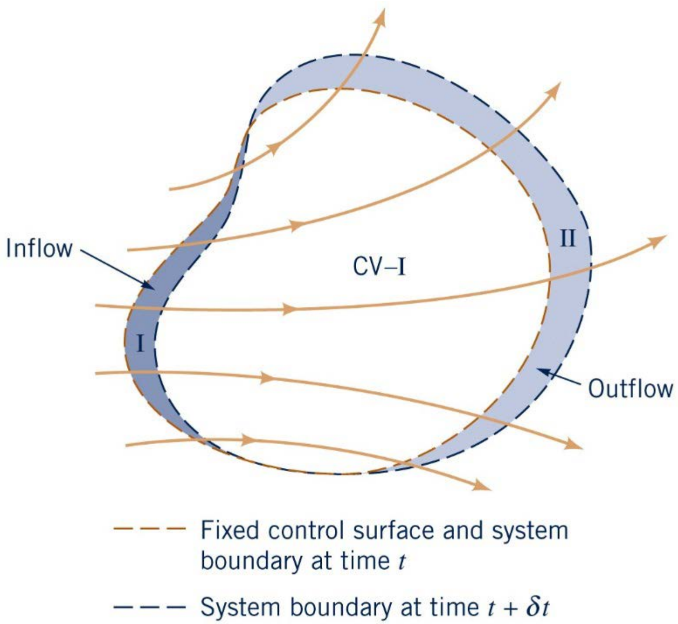
RTT for Arbitrary Fixed CV



Control volume (CV) and system for flow through an arbitrary, fixed control volume

B	$b = B/m$
m	1
$m\mathbf{V}$	$\mathbf{V}$
E	e

The relationship between the time rate of B for a system and that for the control volume is given by

$$\underbrace{\frac{DB_{\text{sys}}}{Dt}}_{\substack{\text{Time rate of}\\ \text{change of B}\\ \text{within a system}}} = \underbrace{\frac{dB_{\text{CV}}}{dt}}_{\substack{\text{Time rate of}\\ \text{chagne of B}\\ \text{within CV}}} + \underbrace{\dot{B}_{\text{out}}}_{\substack{\text{Outflux of B}\\ \text{through CS}}} - \underbrace{\dot{B}_{\text{in}}}_{\substack{\text{Influx of B}\\ \text{through CS}}}$$

For an arbitrary fixed CV,

$$B_{\text{CV}} = \int_{\text{CV}} \beta\rho dV$$

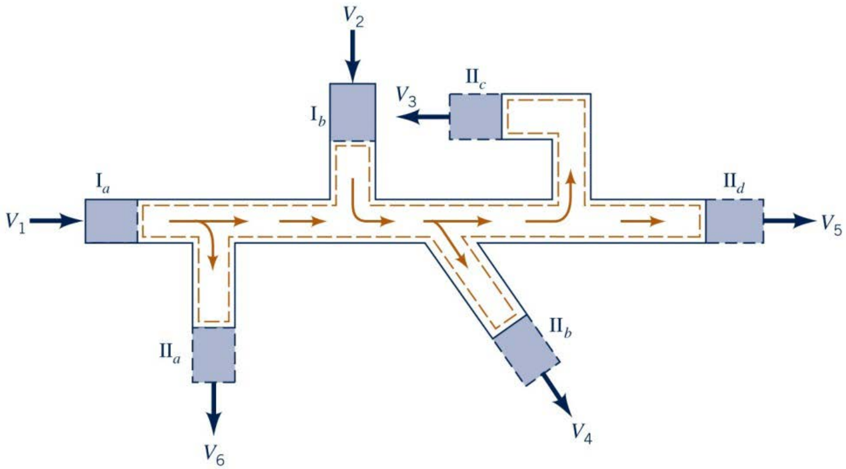
$$\dot{B}_{\text{out}} = \int_{\text{CS}_{\text{out}}} \beta\rho \underline{V} \cdot \underline{n}\, dA$$

$$\dot{B}_{\text{in}} = -\int_{\text{CS}_{\text{in}}} \beta\rho \underline{V} \cdot \underline{n} dA$$

or,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho dV + \int_{\text{CS}} \beta\rho \underline{V} \cdot \underline{n} dA$$

Uniform Flow Across Discrete CS



Typical control volume with more than one inlet and outlet.

At the i^th outlet,

$$\dot{B}_{\text{out},i} = \int_{\text{CS}_{\text{out},i}} \beta\rho \underline{V_i} \cdot \underline{n}_i dA = \beta_i \rho_i V_i A_i$$

At the j^th inlet,

$$\dot{B}_{\text{in},j} = \int_{\text{CS}_{\text{in},j}} \beta\rho \underline{V_j} \cdot \underline{n}_j dA = \beta_j \rho_j V_j A_j$$

where, $V = |\underline{V}|$.

Thus, the surface integrals for the flux terms in RTT can be replaced with simple summations at the inlets and outlets,

$$\frac{DB_{\text{sys}}}{Dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\beta\rho)d\forall + \sum_i (\beta_i \rho_i V_i A_i)_{\text{out}} - \sum_j \left(\beta_j \rho_j V_j A_j\right)_{\text{in}}$$

or

$$\frac{DB_{\text{sys}}}{Dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\beta\rho)d\forall + \sum_i (\beta_i \dot{m}_i)_{\text{out}} - \sum_j \left(\beta_j \dot{m}_j\right)_{\text{in}}$$

where, , $\dot{m} = \rho VA = \rho Q$, and $Q = VA$

Leibniz Integral Rule

Leibnitz theorem allows differentiation of an integral of which limits of integration are functions of the variable (the time t for our case) with which you need to differentiate. For 1D,

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} dx + f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t)$$

As a special case, if $a(t)$ and $b(t)$ are fixed values, e.g., constants x_0 and x_1, respectively,

$$\frac{d}{dt}\int_{x_0}^{x_1} f(x,t)dx = \int_{x_0}^{x_1} \frac{\partial f}{\partial t} dx$$

Thus, for a fixed CV the RTT can also be written as

$$\therefore \frac{DB_{\mathrm{sys}}}{Dt} = \int_{\mathrm{CV}} \frac{\partial}{\partial t}(\beta\rho)d\forall + \int_{\mathrm{CS}} \beta\rho \underline{V}\cdot\underline{n}dA$$

Steady Effects

For a steady flow,

$$\frac{\partial(\quad)}{\partial t} \equiv 0$$

Thus, the RTT can be simplified as

$$\frac{DB_{\text{sys}}}{Dt} = \int_{\text{CV}} \cancel{\frac{\partial}{\partial t}} (\beta\rho) d\mathcal{V} + \int_{\text{CS}} \beta\rho \underline{V} \cdot \underline{n} dA = \int_{\text{CS}} \beta\rho \underline{V} \cdot \underline{n} dA$$

Which indicate that for steady flows the amount of B within the CV does not change with time. If the flow is uniform across discrete CS's,

$$\frac{DB_{\text{sys}}}{Dt} = \sum_i (\beta_i \dot{m}_i)_{\text{out}} - \sum_j (\beta_j \dot{m}_j)_{\text{in}}$$

Gauss's Theorem

Suppose $\mathcal{V}$ is a volume in 3D space and has a piecewise smooth boundary S. If $\underline{F}$ is a continuously differentiable vector field defined on a neighborhood of $\mathcal{V}$, then

$$\int_S \underline{F} \cdot \underline{n} dS = \int_{\mathcal{V}} \nabla \cdot \underline{F} \, d\mathcal{V}$$

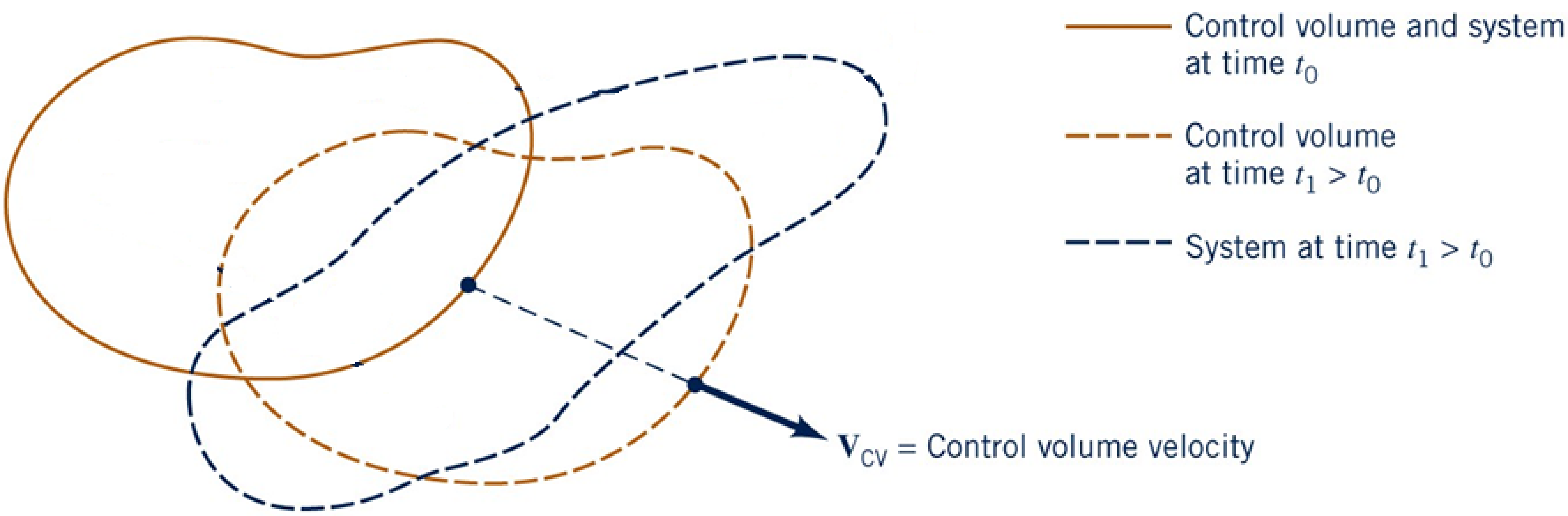
This equation is also known as the 'Divergence theorem.' Thus, the two integral terms in the RTT for a fixed CV can be combined into a single volume integral such that,

$$\frac{DB_{\mathrm{sys}}}{Dt} = \int_{\mathrm{CV}} \left[\frac{\partial}{\partial t} (\beta\rho) + \nabla \cdot \left(\beta\rho\underline{V} \right) \right] d\mathcal{V}$$

This form of RTT will be used in Chapter 6 Differential Analysis.

Moving CV

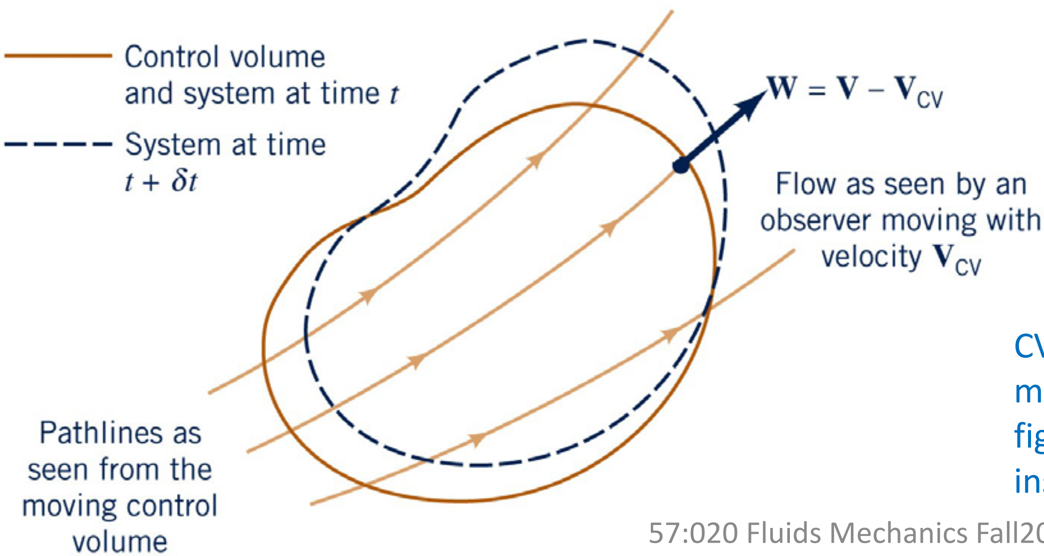
- For most fluids problems, the CV may be considered as a fixed volume. There are, however, situations for which the analysis is simplified if the CV is allowed to move (or deform).
- We consider a CV that moves with a constant velocity $\mathbf{V}_{CV}$ without changes in its shape, size, and orientation with time.



RTT for Moving CV – Contd.

- For a moving (but not deforming) CV, the only difference that needs to be considered is that fact that relative to the moving CV the fluid velocity observed is the relative velocity $\underline{V}_r = \underline{V} - \underline{V}_{\text{CV}}$, not the absolute velocity $\underline{V}$. (Note, $\mathbf{W}$ is used to denote $\underline{V}_r$ in out text book.)
- Thus, the RTT for a moving CV with constant velocity is given by

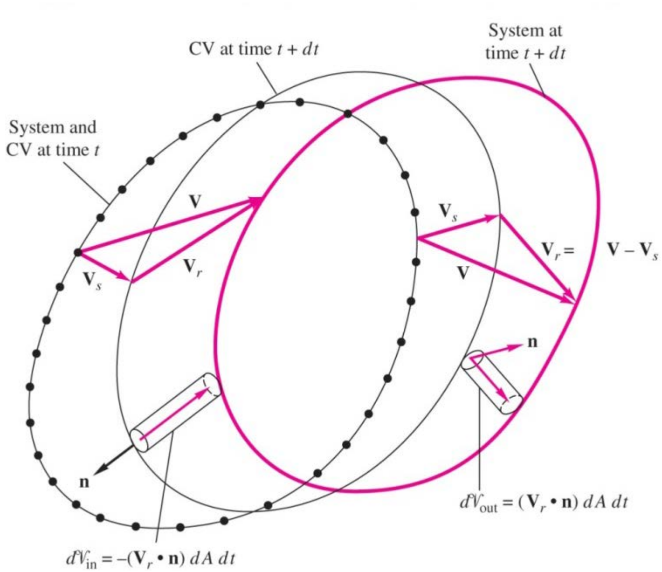
$$\frac{DB_{\text{sys}}}{Dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho dV + \int_{\text{CS}} \beta\rho \underline{V}_r \cdot \underline{n} dA$$



CV and system as seen by an observer moving with the CV. Note that, in this figure, the relative velocity is denoted by $\mathbf{W}$ instead of $\underline{V}_r$.

RTT for Moving and Deforming CV

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho dV + \int_{CS} \beta\rho(\boldsymbol{V_r}\cdot\hat{\boldsymbol{n}})dA^{*}$$



The most general case where both CV and CS change their shape and location with time

$\boldsymbol{V}_r = \boldsymbol{V}(\boldsymbol{x},t) - \boldsymbol{V}_S(\boldsymbol{x},t)$

- $V_S(x,t)$: Velocity of CS
- $V(x,t)$: Fluid velocity in the coordinate system in which the V_s is observed
- V_r: Relative velocity of fluid seen by an observer riding on the CV

*Ref) *Fluid Mechanics* by Frank M. White, McGraw Hill

Example 1

4.68 The wind blows across a field with an approximate velocity profile as shown in Fig. P4.73. Use Eq. 4.16 with the parameter b equal to the velocity to determine the momentum flowrate across the vertical surface A–B, which is of unit depth into the paper.

$$\dot{B}_{\text{out}} = \int_{CS_{\text{out}}} \rho b \underline{V} \cdot \hat{\boldsymbol{n}} dA \quad (4.16)$$

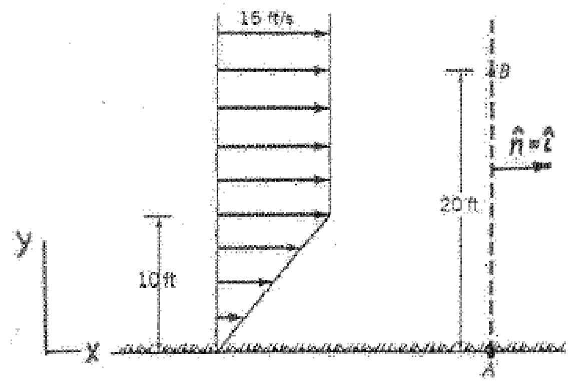


FIGURE P4.73

For momentum $B = m\underline{V}$, the intensive parameter $b(\text{or } \beta) = B/m = \underline{V}$. Thus, for $\text{CS}_{\text{out}} = AB$ of unit depth,

$$\dot{B}_{\text{out}} = \int_{AB} \rho \underline{V}\, \underline{V} \cdot \hat{\boldsymbol{n}}\, dA$$

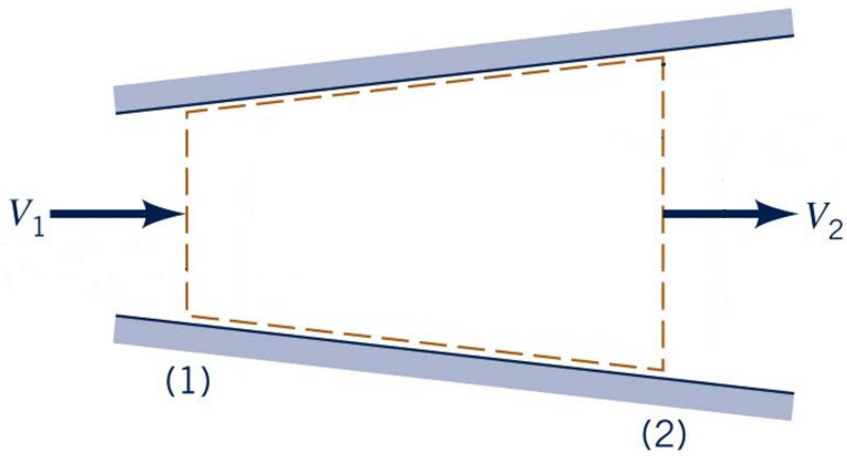
where, $\underline{V} = \left(\frac{15}{10}\right) y\hat{\boldsymbol{\imath}}$ for $0 \le y \le 10$ and $\underline{V} = 15\hat{\boldsymbol{\imath}}$ for $10 \le y \le 20$ and ρ = 0.00238 slugs/ft^3. Thus,

$$\dot{B}_{\text{out}} = \int_0^{10} \rho \left(\frac{15}{10} y\hat{\boldsymbol{\imath}}\right)\left(\frac{15}{10} y\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}\right)(1)dy + \int_{10}^{20} \rho(15\hat{\boldsymbol{\imath}})(15\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}})(1)dy$$

$$= \rho\hat{\boldsymbol{\imath}}\left[\int_0^{10}\left(\frac{15}{10}y\right)^2 dy + \int_{10}^{20}(15)^2 dy\right] = (0.00238)\hat{\boldsymbol{\imath}}\left[\frac{225}{100}\cdot\frac{y^3}{3}\bigg|_0^{10} + 225y\bigg|_{10}^{20}\right] = 7.14\hat{\boldsymbol{\imath}} \text{ slug} \cdot \text{ft/s}^2$$

$$= 7.14\hat{\boldsymbol{\imath}} \text{ lbf}$$

Example 2



Given:

- Water flow ($\rho = \text{constant}$)
- D_1 = 10 cm; D_2 = 15 cm
- V_1 = 10 cm/s
- Steady flow

Find: V_2 to satisfy the mass conservation?

RTT for fixed CV:

$$\frac{DB_{\text{sys}}}{Dt} = \int_{\text{CV}} \frac{\partial}{\partial t}(\beta\rho)d\forall + \int_{\text{CS}} \beta\rho \underline{V} \cdot \underline{n} dA$$

For the mass conservation , $B = m$ and $\beta = 1$,

Steady flow

$$\therefore \frac{Dm_{\text{sys}}}{Dt} = 0 = \int_{\text{CV}} \frac{\partial\rho}{\partial t} d\forall + \int_{\text{CS}} \rho \underline{V} \cdot \underline{n} dA$$

Example 2 – Contd.

Also, since the flow is uniform across discrete CS,

$$\frac{DB_{sys}}{Dt} = \sum_i (\beta_i \dot{m}_i)_{\text{out}} - \sum_j (\beta_j \dot{m}_j)_{\text{in}}$$

with $B = m$ and $\beta = 1$ for one outlet and one inlet,

$$0 = m_2 - m_1$$

or

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

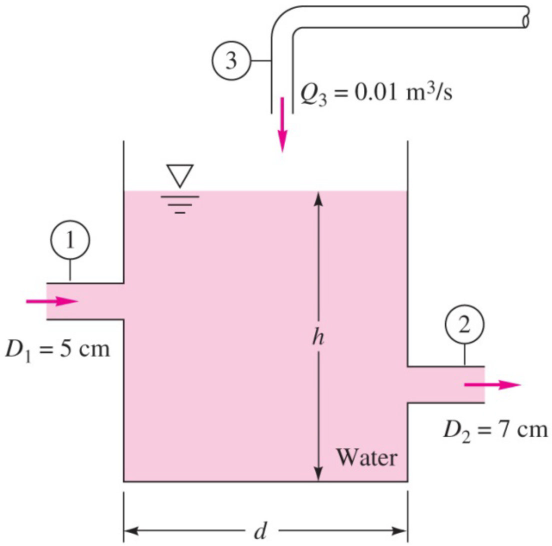
Since $\rho_1 = \rho_2$,

$$V_1 A_1 = V_2 A_2$$

Thus,

$$V_2 = \frac{A_1}{A_2} V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{10\text{ cm}}{15\text{ cm}}\right)^2 (10\ ) = 4.4\text{ cm/s}$$

Example 3



Given:

- D_1 = 5 cm; D_2 = 7 cm
- V_1 = 3 m/s
- $Q_3 = V_3 A_3 = 0.01$ m^3/s
- h = constant (i.e., steady flow)
- $\rho_1 = \rho_2 = \rho_3 = \rho$ for water (incompressible)

Find: V_2 to satisfy the mass conservation ?

RTT for a steady and uniform flow across discrete CS:

$$0 = \sum_i (\dot{m}_i)_{\text{out}} - \sum_j (\dot{m}_j)_{\text{in}}$$

where, $\dot{m} = \rho Q = \rho V A$. With one outlet and two inlets,

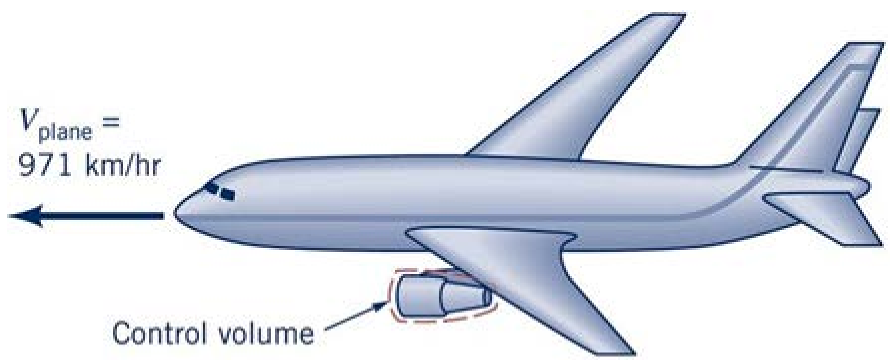
$$0 = \rho V_2 A_2 - \rho V_1 A_1 - \rho Q_3$$

By solving for V_2,

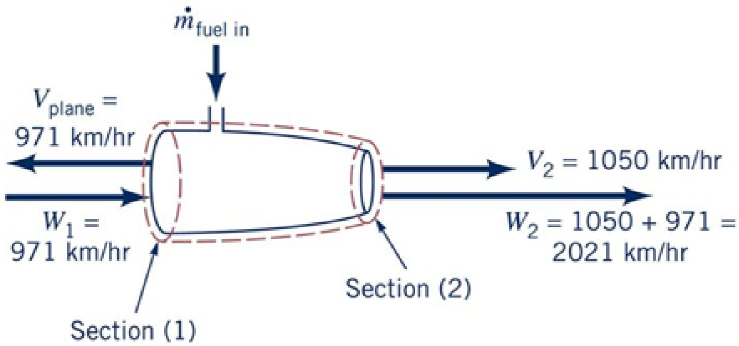
$$V_2 = \frac{V_1 A_1 + Q_3}{A_2} = \frac{(3)(\pi)(0.05)^2/4 + (0.01)}{(\pi)(0.07)^2/4} = 4.13 \text{ m/s}$$

Example 4

An airplane moves forward at a speed of 971 km/hr. The front area of the jet engine is 0.80 m^2 and the entering air density is 0.736 kg/m^3. A stationary observer determines that relative to the Earth, the jet engine exhaust gases move away from the engine with a speed of 1050 km/hr. The engine exhaust area is 0.558 m^2, and the exhaust gas density is 0.515 kg/m^3. Estimate the mass flowrate of fuel into the engine in kg/hr.



Example 4 – Contd.



$$\frac{DB_{\text{sys}}}{Dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho dV + \int_{\text{CS}} \beta\rho \underline{V_r} \cdot \underline{n} dA$$

Assuming 1D flow,

$$-\dot{m}_{\text{fuel}} - \rho_1 A_1 V_{r1} + \rho_2 A_2 V_{r2} = 0$$

or

$$\dot{m}_{\text{fuel}} = \rho_2 A_2 V_{r2} - \rho_1 A_1 V_{r1}$$

Since

$$V_{r1} = V_1 - V_{\text{plane}} = 0 - (-971) = 971 \text{ km/hr}$$
$$V_{r2} = V_2 - V_{\text{plane}} = 1050 - (-971) = 2021 \text{ km/hr}$$

Thus,

$$\dot{m}_{\text{fuel}}$$
$$= (0.515)(0.558)(2021)(1000\,\text{m/km})$$
$$- (0.515)(0.558)(2021)(1000\,\text{m/km}) = (580{,}800 - 571{,}700)$$
$$= 9100\,\text{kg/hr}$$

Continuity Equation (Ch. 5.1)

RTT with B = mass and β = 1,

$$\underbrace{0 = \frac{Dm_{\text{sys}}}{Dt}}_{\text{mass conservatoin}} = \frac{d}{dt}\int_{\text{CV}} \rho d\forall + \int_{\text{CS}} \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA$$

or

$$\underbrace{\int_{\text{CS}} \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA}_{\substack{\text{Net rate of outflow} \\ \text{of mass across CS}}} = \underbrace{-\frac{d}{dt}\int_{\text{CV}} \rho d\forall}_{\substack{\text{Rate of decrease of} \\ \text{mass within CV}}}$$

Note: Incompressible fluid (ρ = constant)

$$\int_{CS} \underline{V} \cdot \hat{\boldsymbol{n}} dA = -\frac{d}{dt}\int_{CV} d\forall$$

(Conservation of volume)

Simplifications

1. Steady flow

$$\int_{\text{CS}} \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA = 0$$

2. If $\underline{V}$ = constant over discrete CS's (i.e., one-dimensional flow)

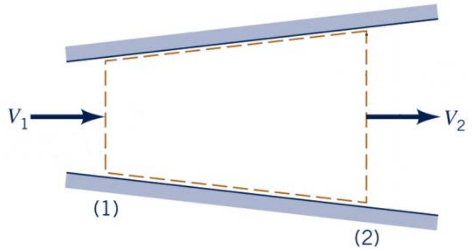
$$\int_{\text{CS}} \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA = \sum_{\text{out}} \rho V A - \sum_{\text{in}} \rho V A$$

3. Steady one-dimensional flow in a conduit

$$(\rho V A)_{\text{out}} - (\rho V A)_{\text{in}} = 0$$

or

$$\rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0$$



For ρ = constant

$$V_1 A_1 = V_2 A_2 \quad (\text{or } Q_1 = Q_2)$$

Some useful definitions

- Mass flux (or mass flow rate) $\dot{m} = \int_A \rho \underline{V} \cdot \underline{dA}$ $\quad (= \rho V A \text{ for uniform flow})$

- Volume flux (flow rate) $Q = \int_A \underline{V} \cdot \underline{dA}$ $\quad (= V A \text{ for uniform flow})$

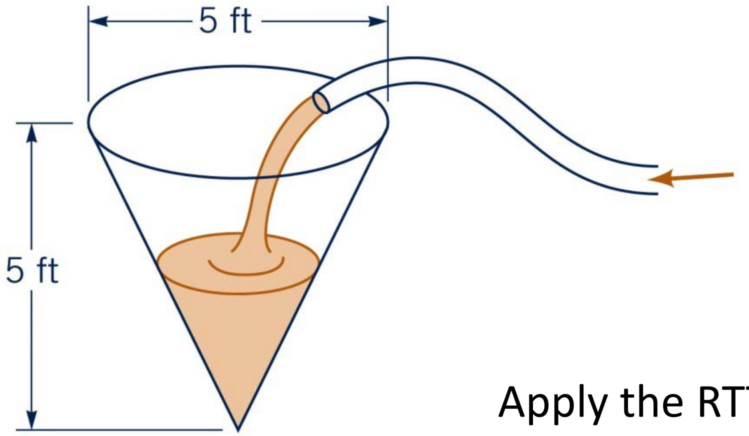
Note: $\underline{dA} = \hat{\boldsymbol{n}} dA$

- Average velocity $\bar{A} = \frac{Q}{A} = \frac{1}{A} \int_A \underline{V} \cdot \underline{dA}$

- Average density $\bar{\rho} = \frac{1}{A} \int_A \rho dA$

Note: $\dot{m} \neq \bar{\rho} Q$ unless ρ = constant

Example 5



Estimate the time required to fill with water a cone-shaped container 5 ft hight and 5 ft across at the top if the filling rate is 20 gal/min.

Apply the RTT for conservation of mass, i.e., β = 1

$$0 = \frac{d}{dt}\int_{\mathrm{CV}} \rho d\mathcal{V} + \int_{\mathrm{CS}} \rho \underline{V} \cdot \hat{\boldsymbol{n}} dA$$

For incompressible fluid (i.e., ρ = constant) and one inlet,

$$0 = \frac{d}{dt}\underbrace{\int_{\mathrm{CV}} d\mathcal{V}}_{=\mathcal{V}(t)} - \underbrace{(VA)_{\mathrm{in}}}_{=Q}$$

Example 4 – Contd.

Volume of the cone at time t,

$$\forall(t) = \frac{\pi D^2}{12} h(t)$$

Flow rate at the inlet,

$$Q = \left(20\,\frac{\text{gal}}{\text{min}}\right)\left(231\,\frac{\text{in}^3}{\text{gal}}\right) \Big/ \left(1{,}728\,\frac{\text{in}^3}{\text{ft}^3}\right) = 2.674\ \text{ft}^3/\text{min}$$

The continuity eq. becomes

$$0 = \frac{d}{dt}\left(\frac{\pi D^2}{12} \cdot h\right) - Q$$

or

$$\frac{dh}{dt} = \frac{12Q}{\pi D^2} \qquad (1)$$

Example 4 – Contd.

Solve the 1st order ODE for $h(t)$,

$$h(t) = \int_0^t \frac{12Q}{\pi D^2} dt = \frac{12Q \cdot t}{\pi D^2}$$

Thus, the time for h = 5 ft is

$$t = \frac{\pi D^2 h}{12Q} = \frac{\pi (5\ \text{ft})^2 (5\ \text{ft})}{(12)(2.674\ \text{ft}^3/\text{min})} = 12.2\ \text{min}$$