Review for Exam2

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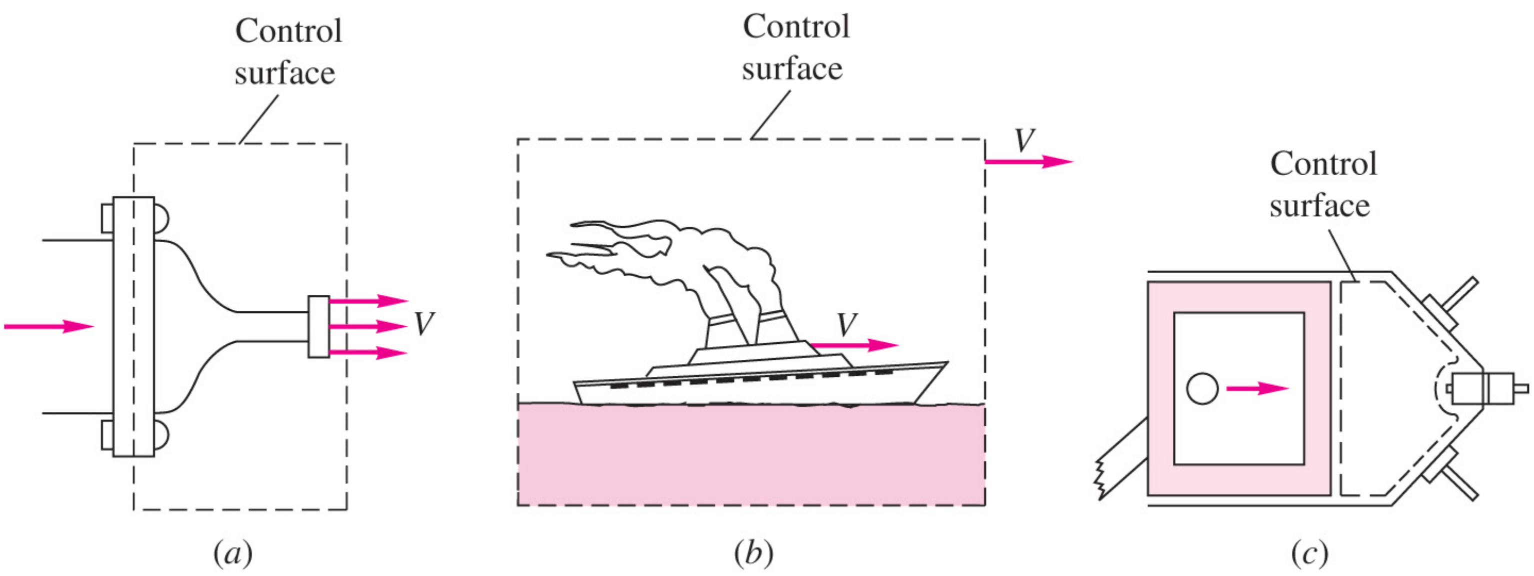
System vs. Control volume

- **System**: A collection of real matter of fixed identity.
- **Control volume (CV)**: A geometric or an imaginary volume in space through which fluid may flow. A CV may move or deform.



Laws of Mechanics for a System

Laws of mechanics are written for a system, i.e., for a fixed amount of matter

- Conservation of mass

$$\frac{Dm_{\text{sys}}}{Dt} = 0$$

- Conservation of momentum

$$\frac{D\left(m\underline{V}\right)_{\text{sys}}}{Dt} = \underline{F}$$

- Conservation of energy

$$\frac{DE_{\text{sys}}}{Dt} = \dot{Q} - \dot{W}$$

Governing Differential Eq. (GDE):

$$\therefore \frac{D}{Dt} \underbrace{\left(m, m\underline{V}, E\right)}_{\substack{\text{system extensive} \\ \text{properties, } B_{\text{sys}}}} = \text{RHS}$$

Reynolds Transport Theorem (RTT)

- In fluid mechanics, we are usually interested in a region of space, i.e., CV and not particular systems. Therefore, we need to transform GDE's from a system to a CV, which is accomplished through the use of RTT

$$\underbrace{\frac{DB_{\text{sys}}}{Dt}}_{\substack{\text{time rate of change} \\ \text{of } B \text{ for a system}}} = \underbrace{\frac{D}{Dt}\int_{\text{CV}(\underline{x},t)} \beta\rho d\forall}_{\substack{\text{time rate of change} \\ \text{of } B \text{ in CV}}} + \underbrace{\int_{\text{CS}(\underline{x},t)} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net flux of } B \\ \text{across CS}}}$$

where, $\beta = \frac{dB}{dm} = \left(1, \underline{V}, e\right)$ for $B = (m, m\underline{V}, E)$

- Fixed CV,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho d\forall + \int_{\text{CS}} \beta\rho \underline{V} \cdot d\underline{A}$$

Note:

$$B_{\text{CV}} = \int_{\text{CV}} \beta dm = \int_{\text{CV}} \beta\rho d\forall$$

$$\dot{B}_{\text{CS}} = \int_{\text{CS}} \beta d\dot{m} = \int_{\text{CS}} \beta\rho \underline{V} \cdot d\underline{A}$$

Continuity Equation

- RTT with $B = m$ and $\beta = 1$,

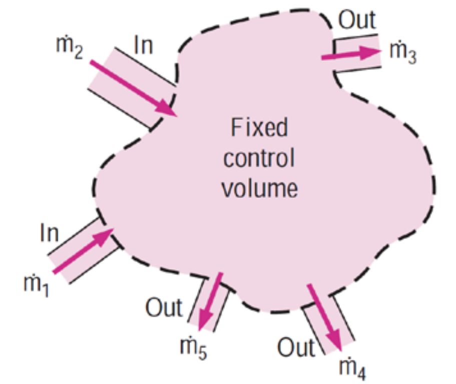
$$\frac{\partial}{\partial t}\int_{CV}\rho d\forall + \int_{CS}\rho \underline{V}\cdot d\underline{A} = 0$$

- Steady flow,

$$\int_{CS}\rho \underline{V}\cdot d\underline{A} = 0$$

- Simplified form,

$$\sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}} = 0$$



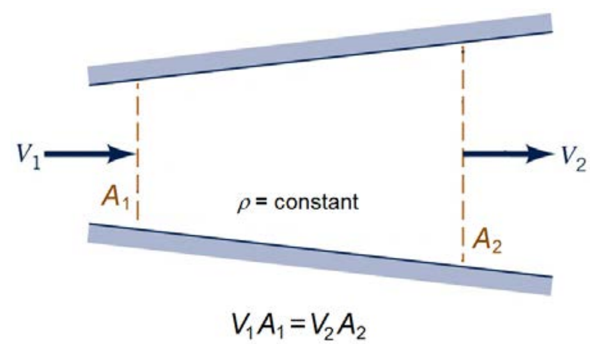
Note: $\dot{m} = \rho Q = \rho V A$

- Conduit flow with one inlet (1) and one outlet (2):

$$\rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0$$

If ρ = constant,

$$V_1 A_1 = V_2 A_2$$



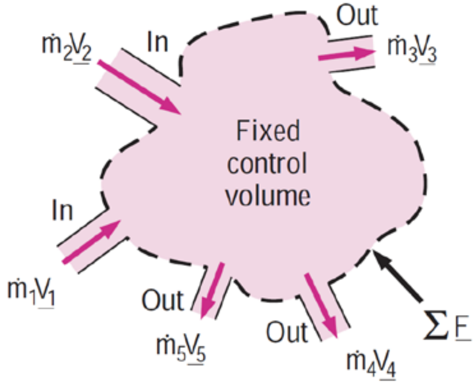
Momentum Equation

- RTT with $B = m\underline{V}$ and $\beta = \underline{V}$,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \underline{V}\rho d\forall + \int_{\mathrm{CS}} \underline{V}\rho \underline{V}\cdot d\underline{A} = \sum \underline{F}$$

- Simplified form:

$$\sum(\dot{m}\underline{V})_{\mathrm{out}} - \sum(\dot{m}\underline{V})_{\mathrm{in}} = \sum \underline{F}$$



or in component forms,

$$\begin{aligned}
\sum(\dot{m}u)_{\mathrm{out}} - \sum(\dot{m}u)_{\mathrm{in}} &= \sum F_x \\
\sum(\dot{m}v)_{\mathrm{out}} - \sum(\dot{m}v)_{\mathrm{in}} &= \sum F_y \\
\sum(\dot{m}w)_{\mathrm{out}} - \sum(\dot{m}w)_{\mathrm{in}} &= \sum F_z
\end{aligned}$$

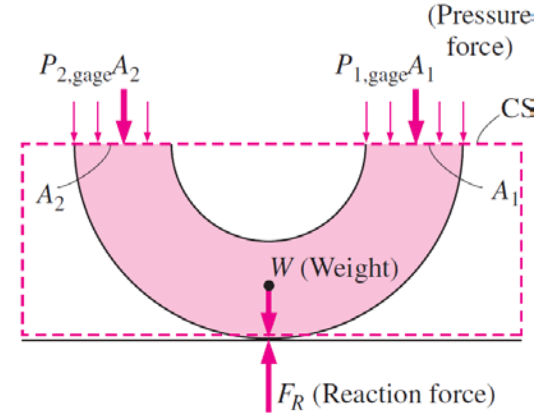
Note: If $\underline{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ is normal to CS, $\dot{m} = \rho V A$, where $V = |\underline{V}|$.

Momentum Equation – Contd.

- External forces:

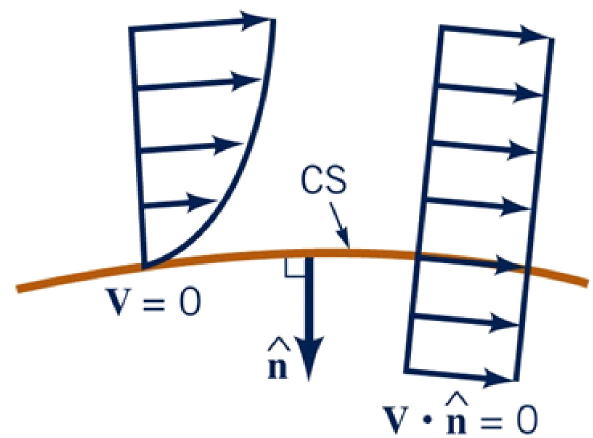
$$\sum \underline{F} = \sum \underline{F}_{\text{body}} + \sum \underline{F}_{\text{surface}} + \sum \underline{F}_{\text{other}}$$

  - $\sum \underline{F}_{\text{body}} = \sum \underline{F}_{\text{gravity}}$
    - $\sum \underline{F}_{\text{gravity}}$: gravity force (i.e., weight)
  - $\sum \underline{F}_{\text{Surface}} = \sum \underline{F}_{\text{pressure}} + \sum \underline{F}_{\text{friction}}$
    - $\sum \underline{F}_{\text{pressure}}$: pressure forces normal to CS
    - $\sum \underline{F}_{\text{friction}}$: viscous friction forces tangent to CS
  - $\sum \underline{F}_{\text{other}}$: anchoring forces or reaction forces



An 180° elbow supported by the ground

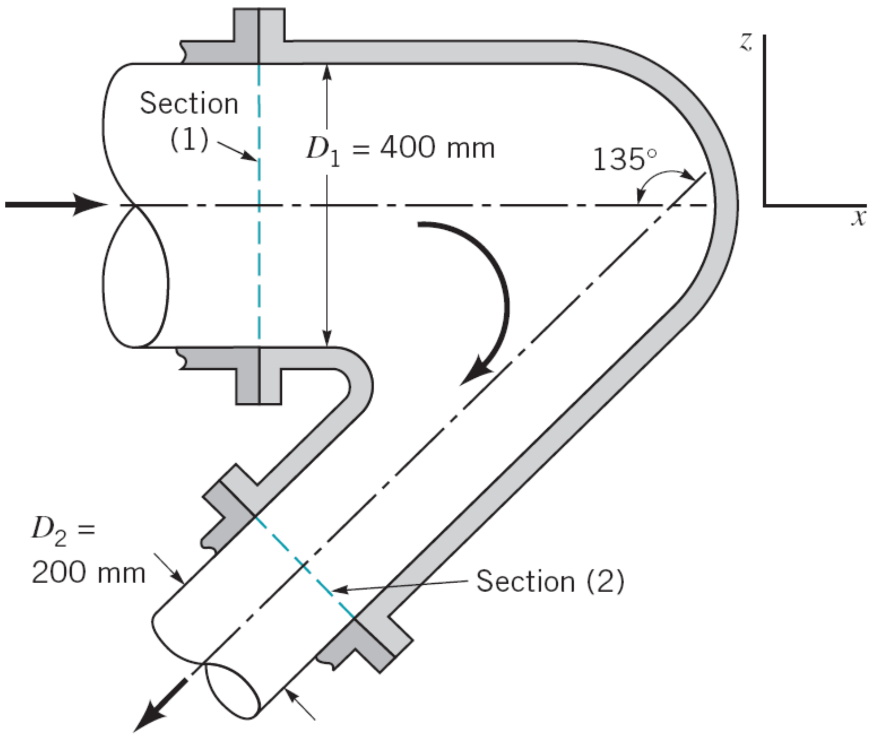
In most flow systems, the force $\vec{F}$ consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.



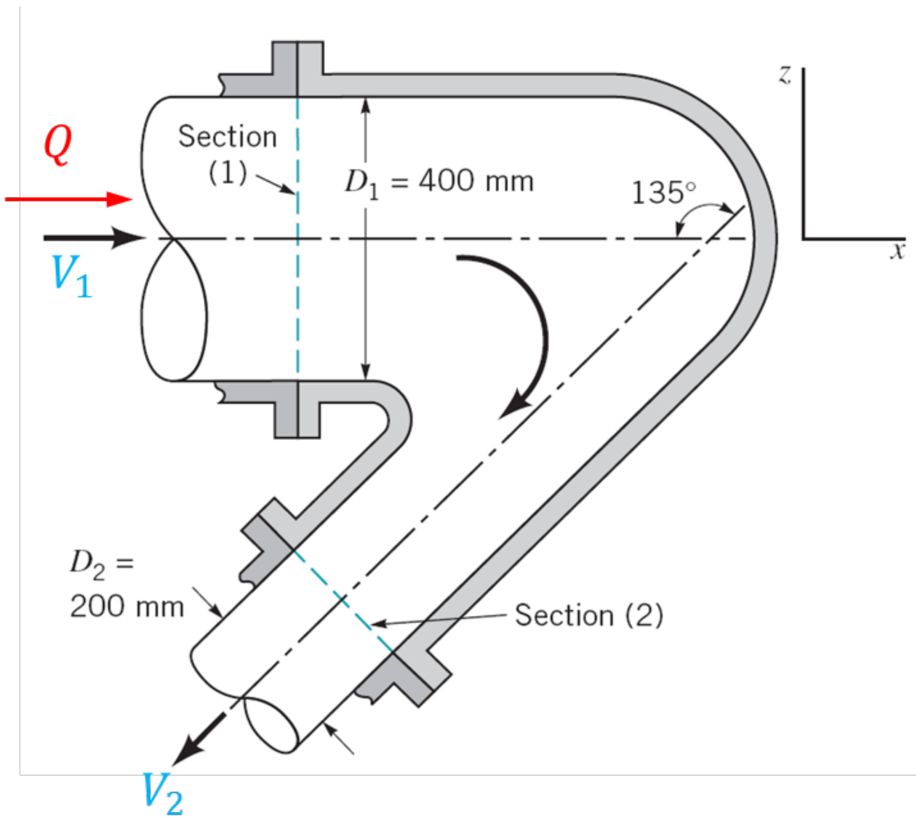
Note: Shearing forces can be avoided by carefully selecting the CV such that CS's are parallel with the flow direction.

Example (Bend)

5.34 A converging elbow (see Fig. P5.34) turns water through an angle of 135° in a vertical plane. The flow cross section diameter is 400 mm at the elbow inlet, section (1), and 200 mm at the elbow outlet, section (2). The elbow flow passage volume is 0.2 m³ between sections (1) and (2). The water volume flowrate is 0.4 m³/s and the elbow inlet and outlet pressures are 150 kPa and 90 kPa. The elbow mass is 12 kg. Calculate the horizontal (x direction) and vertical (z direction) anchoring forces required to hold the elbow in place.



Example (Bend) – Contd.

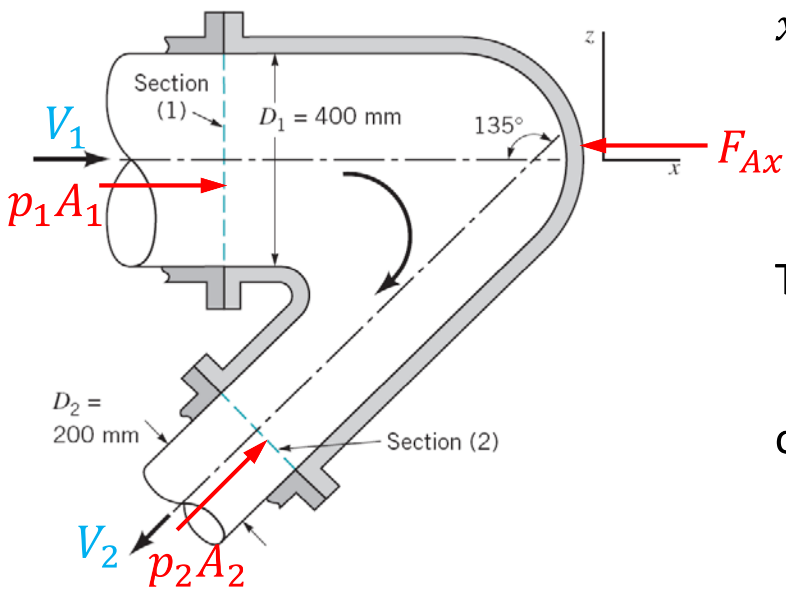


$$Q = 0.4 \text{ m}^3/\text{s}$$
$$D_1 = 0.4 \text{ m}$$
$$D_2 = 0.2 \text{ m}$$

$$V_1 = \frac{Q}{A_1} = \frac{Q}{\pi D_1^2/4} = \frac{0.4}{\pi(0.4)^2/4} = 3.18 \text{ m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi D_2^2/4} = \frac{0.4}{\pi(0.2)^2/4} = 12.73 \text{ m/s}$$

Example (Bend) – Contd.



x-momentum:

$$\sum F_x = (\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}}$$

Thus,

$$-F_{Ax} + p_1A_1 + p_2A_2\cos 45^\circ = (\rho Q)(-V_2\cos 45^\circ) - (\rho Q)(V_1)$$

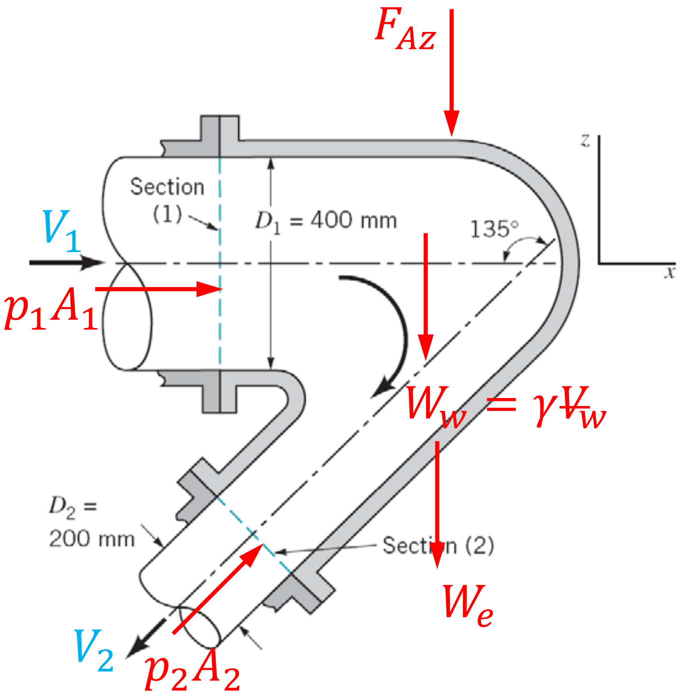
or

$$F_{Ax} = p_1A_1 + p_2A_2\cos 45^\circ + (\rho Q)(V_1 + V_2\cos 45^\circ)$$

$$= (150{,}000)\frac{\pi(0.4)^2}{4} + (50{,}000)\frac{\pi(0.2)^2}{4}\cos 45^\circ + (999)(4)(3.18 + 12.73\cos 45^\circ)$$

$$\therefore F_{Ax} = 25{,}700\ \text{N}$$

Example (Bend) – Contd.



z-momentum:

$$\sum F_z = (\dot{m}w)_{\text{out}} - (\dot{m}w)_{\text{in}}$$

Thus,

$$-F_{Az} + p_2A_2 \sin 45^\circ - W_w - W_e$$
$$= (\rho Q)(-V_2 \sin 45^\circ) - (\rho Q)(0)$$

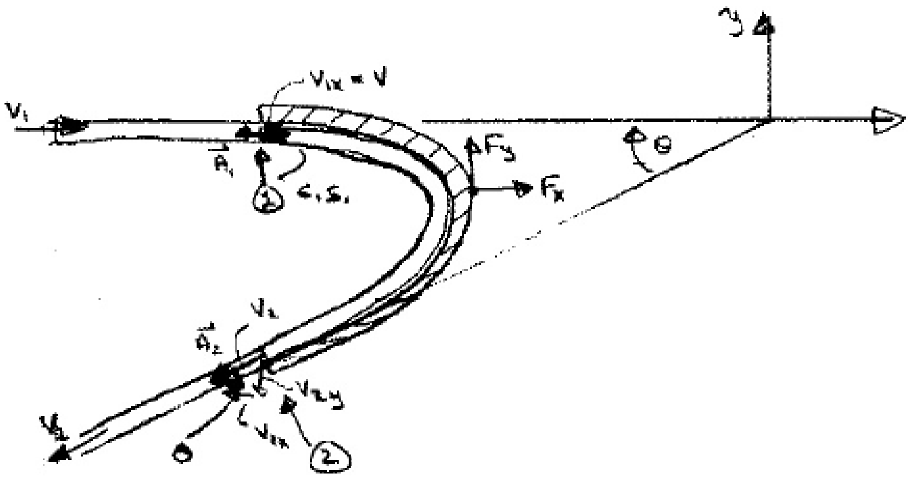
or

$$F_{Az} = p_2A_2 \cos 45^\circ - \gamma V\!\!\!/_w - W_e + (\rho Q)(V_2 \sin 45^\circ)$$

$$= (50{,}000)\frac{\pi(0.2)^2}{4}\sin 45^\circ - (9800)(0.2) - (12)(9.81)$$
$$+ (999)(4)(12.73 \sin 45^\circ)$$

$$\therefore F_{Ax} = 8{,}920 \text{ N}$$

Typical Example (1): Vane



Energy eq.:

$$p_1 + \frac{1}{2}\rho V_1^2 + z_1 = p_2 + \frac{1}{2}\rho V_2^2 + z_2 + h_L$$

with $p_1 = p_2 = 0$, $z_1 \approx z_2$, and $h_L \approx 0$,

$$\therefore V_1 = V_2 = V_j$$

Continuity:

$$V_1 A_1 = V_2 A_2 = V_j A_j \quad \Rightarrow \quad \dot{m} = \rho V_j A_j$$

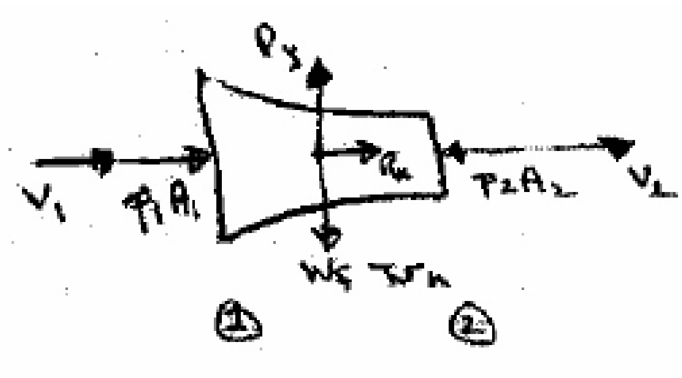
x-momentum:

$$F_x = \underbrace{\dot{m}(-V_2 \cos\theta)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$F_y - W_{\text{fluid}} - W_{\text{vane}} = \underbrace{\dot{m}(-V_2 \sin\theta)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Typical Example (2): Nozzle



Continuity:

$$V_1 A_1 = V_2 A_2$$

$$\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$$

Energy eq. with $p_2 = 0$ and $z_1 = z_2$:

$$p_1 + \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_2^2 + h_L$$

x-momentum:

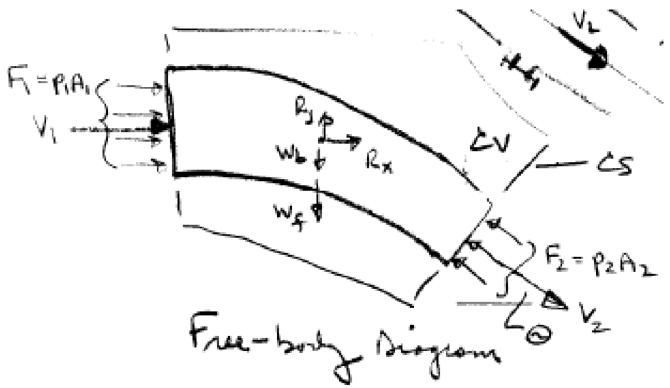
$$R_x + p_1 A_1 = \underbrace{\dot{m}(V_2)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$R_y - W_{\text{fluid}} - W_{\text{nozzle}} = \underbrace{\dot{m}(0)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Typical Example (3): Bend



Continuity:

$$V_1 A_1 = V_2 A_2$$

$$\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$$

Energy eq.:

$$p_1 + \frac{1}{2}\rho V_1^2 + z_1 = p_2 + \frac{1}{2}\rho V_2^2 + z_2 + h_L$$

x-momentum:

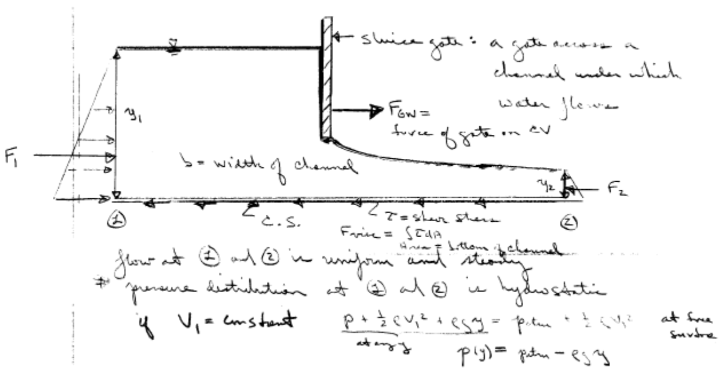
$$R_x + p_1 A_1 - p_2 A_2 \cos\theta = \underbrace{\dot{m}(V_2\cos\theta)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$R_y + p_2 A_2 \sin\theta - W_{\text{fluid}} - W_{\text{bend}} = \underbrace{\dot{m}(-V_2\sin\theta)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Typical Example (4): Sluice gate



Continuity:

$$V_1(y_1 b) = V_2(y_2 b)$$

$$\dot{m} = \rho V_1(y_1 b) = \rho V_2(y_2 b)$$

Energy (Bernoulli) eq. with $p_1 = p_2$ and $h_L \approx 0$:

$$\frac{1}{2}\rho V_1^2 + y_1 = \frac{1}{2}\rho V_2^2 + y_2$$

x-momentum:

$$F_{GW} + \underbrace{\gamma\left(\frac{y_1}{2}\right)(y_1 b)}_{\bar{p}_1 A_1} - \underbrace{\gamma\left(\frac{y_2}{2}\right)(y_2 b)}_{\bar{p}_2 A_2} = \underbrace{\dot{m}(V_2)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$0 = \underbrace{\dot{m}(0)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Energy Equation

- RTT with $B = E$ and $\beta = e$,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} e\rho d\forall + \int_{\mathrm{CS}} e\rho \underline{V}\cdot d\underline{A} = \dot{Q} - \dot{W}$$

- Simplified form:

$$\frac{p_{\mathrm{in}}}{\gamma} + \alpha_{\mathrm{in}}\frac{V_{\mathrm{in}}^2}{2\mathrm{g}} + z_{\mathrm{in}} + h_p = \frac{p_{\mathrm{out}}}{\gamma} + \alpha_{\mathrm{out}}\frac{V_{\mathrm{out}}^2}{2\mathrm{g}} + z_{\mathrm{out}} + h_t + h_L$$

- V in energy equation refers to average velocity $\bar{V}$

- α : kinetic energy correction factor $= \begin{cases} 1 \text{ for uniform flow across CS} \\ 2 \text{ for laminar pipe flow} \\ \approx 1 \text{ for turbulent pipe flow} \end{cases}$

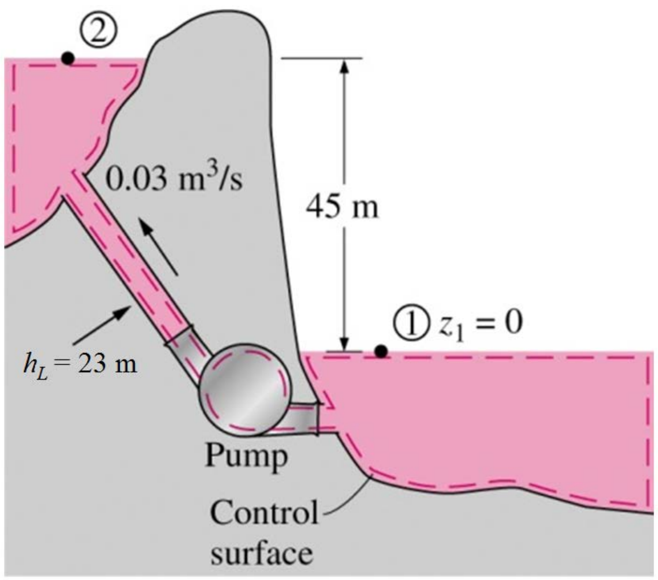
Energy Equation - Contd.

Uniform flow across CS's:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2\text{g}} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2\text{g}} + z_1 + h_t + h_L$$

- Pump head $h_p = \frac{\dot{W}_p}{\dot{m}\text{g}} = \frac{\dot{W}_p}{\rho Q\text{g}} = \frac{\dot{W}_p}{\gamma Q} \Rightarrow \dot{W}_p = \dot{m}\text{g}h_p = \rho\text{g}Qh_p = \gamma Q h_p$
- Turbine head $h_t = \frac{\dot{W}_t}{\dot{m}\text{g}} = \frac{\dot{W}_t}{\rho Q\text{g}} = \frac{\dot{W}_t}{\gamma Q} \Rightarrow \dot{W}_t = \dot{m}\text{g}h_t = \rho\text{g}Qh_t = \gamma Q h_t$
- Head loss $h_L = \text{loss}/\text{g} = (\hat{u}_2 - \hat{u}_1)/\text{g} - \dot{Q}/\dot{m}\text{g} > 0$

Example (Pump)



Energy equation:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

With $p_1 = p_2 = 0$, $V_1 = V_2 \approx 0$, $h_t = 0$, and $h_L = 23$ m

$$h_p = (z_2 - z_1) + h_L = 45 + 23 = 68 \text{ m}$$

Pump power,

$$\dot{W}_p = \gamma Q h_p = \frac{(68)(9790)(0.03)}{746} = 80 \text{ hp}$$

(Note: 1 hp = 746 N·m/s = 550 ft·lbf/s)

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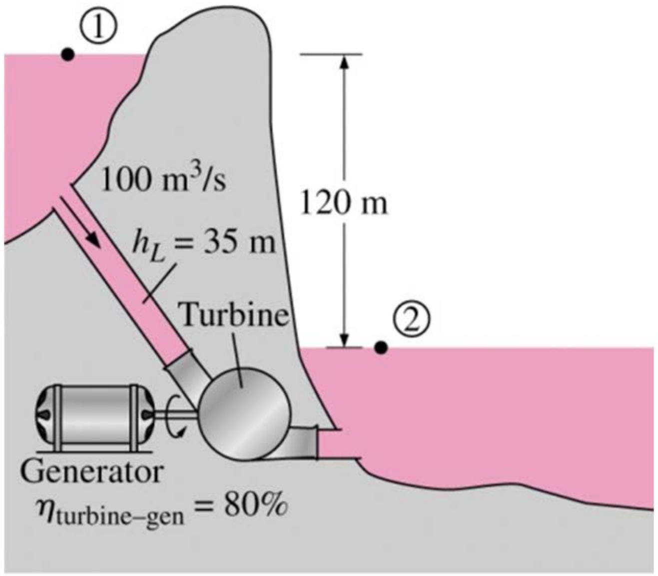
Example (Turbine)



Energy equation:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

With $p_1 = p_2 = 0$, $V_1 = V_2 \approx 0$, $h_p = 0$, and $h_L = 35\ \mathrm{m}$

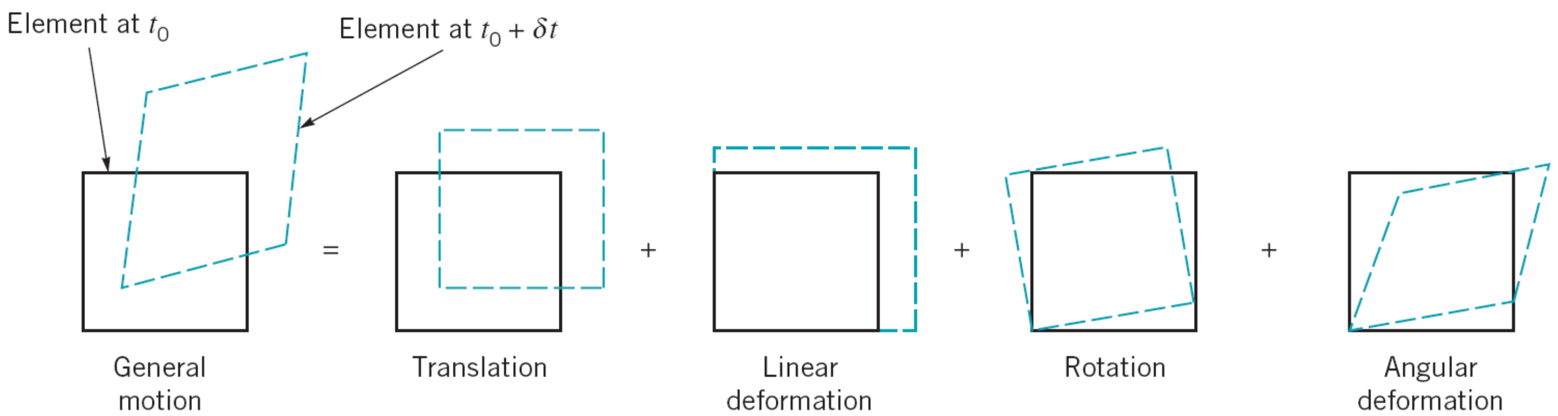
$$h_t = (z_1 - z_2) - h_L = 120 - 35 = 85\ \mathrm{m}$$

Pump power,

$$\dot{W}_t = h_t \gamma Q = (85)(9790)(100) = 83.2\ \mathrm{MW}$$

Differential Analysis

- Fluid Element Kinematics



- Linear deformation(dilatation): $\nabla \cdot \underline{V}$

  $\Rightarrow$ if the fluid is **incompressible** $\qquad \boldsymbol{\nabla} \cdot \underline{\boldsymbol{V}} = \mathbf{0}$

- Rotation(vorticity): $\underline{\xi} = 2\underline{\omega} = \nabla \times \underline{V}$

  $\Rightarrow$ if the fluid is **irrotational** $\qquad \boldsymbol{\nabla} \times \underline{\boldsymbol{V}} = \mathbf{0}$

- Angular deformation is related to shearing stress

  (e.g., $\tau_{ij} = 2\mu\varepsilon_{ij}$ for Newtonian fluids)

Differential Analysis
- Mass Conservation

For a fluid particle,

$$\lim_{\mathrm{CV}\to 0}\left[\int_{\mathrm{CV}}\frac{\partial\rho}{\partial t}\,d\forall+\int_{CS}\rho\underline{V}\cdot d\underline{A}\right]$$

$$=\lim_{\mathrm{CV}\to 0}\int_{\mathrm{CV}}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)\right]d\forall=0$$

$$\therefore\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)=0$$

For an incompressible flow: $\nabla\cdot\underline{V}=0$

Differential Analysis
- Momentum Conservation

$$\lim_{\mathrm{CV}\to 0}\left[\int_{\mathrm{CV}} \frac{\partial \underline{V}}{\partial t}\rho d\forall + \int_{\mathrm{CS}} \underline{V}\rho\underline{V}\cdot\underline{dA}\right] = \sum \underline{F}$$

or

$$\lim_{\mathrm{CV}\to 0}\int_{\mathrm{CV}} \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) d\forall = \sum \underline{F}$$

$$\therefore \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \sum \underline{f} \qquad (\underline{f} = \underline{F} \text{ per unit volume})$$

$$\Rightarrow \underbrace{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right)}_{=\frac{D\underline{V}}{Dt}=\underline{a}} = \underbrace{-\rho g\hat{k}}_{\text{body force due to gravity force}} \quad \underbrace{\underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\nabla\cdot\tau_{ij}}_{\text{viscous shear force}}}_{\text{surface force}}$$

Navier-Stokes Equations

For incompressible, Newtonian fluids,

- Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho \mathrm{g}_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho \mathrm{g}_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho \mathrm{g}_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Solving the NS Eqns

1) Set up the problem and geometry (e.g., sketches), identifying all relevant dimensions and parameters.

2) List all appropriate assumptions, approximations, simplifications, and boundary conditions.

3) Simplify the differential equations of motion (continuity and Navier-Stokes) as much as possible.

4) Integrate the equations, leading to one or more constants of integration

5) Apply boundary conditions to solve for the constants of integration.

6) Verify your results.

Exact Solutions of NS Eqns.

The flow of interest is assumed additionally (than incompressible & Newtonian), for example,

1) Steady (i.e., $\boldsymbol{\partial/\partial t = 0}$ for any variable)
2) Parallel such that the y-component of velocity is zero (i.e., $\boldsymbol{v = 0}$)
3) Purely two dimensional (i.e., $\boldsymbol{w = 0}$ and $\boldsymbol{\partial/\partial z = 0}$ for any velocity component)

e.g.)

$$\frac{\partial u}{\partial x} + \overbrace{\frac{\partial v}{\partial y}}^{2)} + \overbrace{\frac{\partial w}{\partial z}}^{3)} = 0$$

$$\rho\left[\overbrace{\frac{\partial u}{\partial t}}^{1)} + u\overbrace{\frac{\partial u}{\partial x}}^{\text{continuity}} + \overbrace{v}^{2)}\frac{\partial u}{\partial y} + \overbrace{w\frac{\partial u}{\partial z}}^{3)}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\overbrace{\frac{\partial^2 u}{\partial x^2}}^{\text{continuity}} + \frac{\partial^2 u}{\partial y^2} + \overbrace{\frac{\partial^2 u}{\partial z^2}}^{3)}\right]$$

or

$$\therefore \mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x} - \rho g_x$$

Boundary Conditions

Common BC's:

- **No-slip condition** ($\underline{V}_{\text{fluid}} = \underline{V}_{\text{wall}}$; for a stationary wall $\underline{V}_{\text{fluid}} = 0$)
- Interface boundary condition ($\underline{V}_A = \underline{V}_B$ and $\tau_{s,A} = \tau_{s,B}$)
- Free-surface boundary condition ($p_{\text{liquid}} = p_{\text{gas}}$ and $\tau_{s,\text{liquid}} = 0$)
- Symmetry boundary condition

Other BC's:

- Inlet/outlet boundary condition
- Initial condition (for unsteady flow problem)

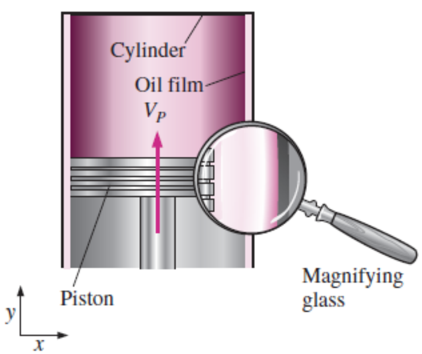


FIGURE 9–51
A piston moving at speed V_P in a cylinder. A thin film of oil is sheared between the piston and the cylinder; a magnified view of the oil film is shown. The *no-slip boundary condition* requires that the velocity of fluid adjacent to a wall equal that of the wall.

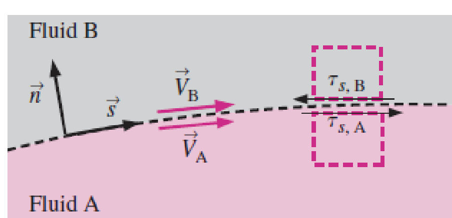


FIGURE 9–52
At an interface between two fluids, the velocity of the two fluids must be equal. In addition, the shear stress parallel to the interface must be the same in both fluids.

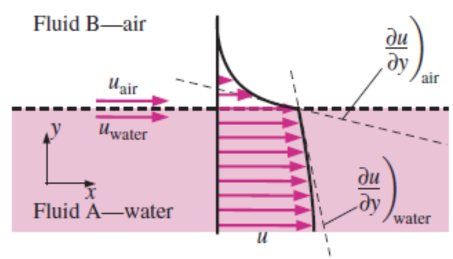


FIGURE 9–53
Along a horizontal *free surface* of water and air, the water and air velocities must be equal and the shear stresses must match. However, since $\mu_{\text{air}} \ll \mu_{\text{water}}$, a good approximation is that the shear stress at the water surface is negligibly small.

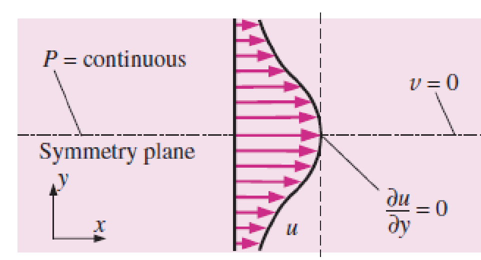
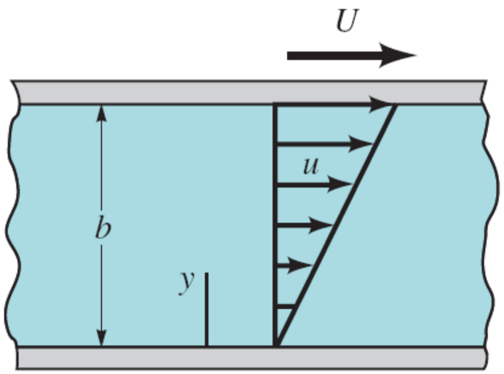


FIGURE 9–54
Boundary conditions along a plane of symmetry are defined so as to ensure that the flow field on one side of the symmetry plane is a *mirror image* of that on the other side, as shown here for a horizontal symmetry plane.

Example: No pressure gradient



$$\mu \frac{d^2u}{dy^2} = 0$$

Integrate twice,

$$u(y) = C_1 y + C_2$$

B.C.,

$$u(0) = (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$u(b) = (C_1)(b) + C_2 = U \quad \Rightarrow \quad C_1 = \frac{U}{b}$$

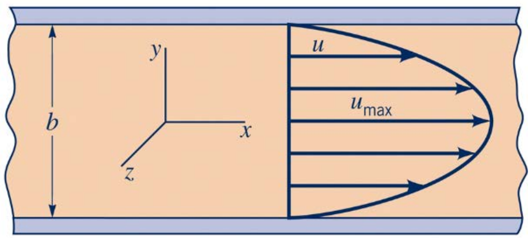
$$\therefore u(y) = \frac{U}{b} y$$

Analysis:

$$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = (\mu)\left(\frac{U}{b}\right) = \frac{\mu U}{b}$$

Example: with Pressure Gradient

Fixed plate



Fixed plate

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Integrate twice,

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

B.C.,

$$u(0) = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)(0)^2 + (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$u(b) = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)(b)^2 + (C_1)(b) + C_2 = 0 \Rightarrow \quad C_1 = -\frac{1}{2\mu}\frac{dp}{dx}b$$

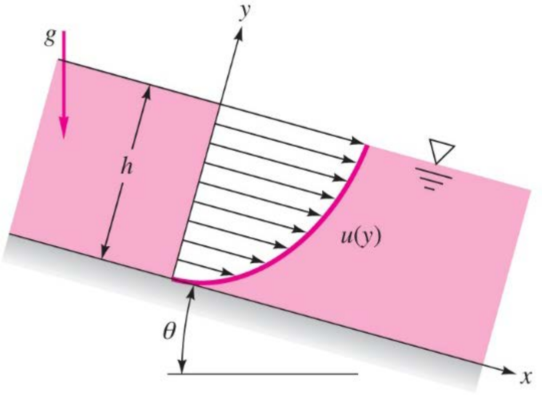
$$\therefore u(y) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)(y^2 - by)$$

Analysis:

$$q = \int_{-h}^{h} u dy = -\frac{b^3}{12\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\tau_w = \mu \frac{du}{dy}\Big)_{y=0} = -\frac{b}{2}\left(\frac{\partial p}{\partial x}\right)$$

Example: Inclined wall



$$\mu \frac{d^2u}{dy^2} = -\rho g_x$$

Integrate twice,

$$u(y) = -\frac{\rho g_x}{2\mu} y^2 + C_1 y + C_2$$

B.C.,

$$u(0) = \left(-\frac{\rho g_x}{\mu}\right)(0)^2 + (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$\left.\frac{du}{dy}\right)_{y=h} = \left(-\frac{\rho g_x}{\mu}\right)(h) + C_1 = 0 \quad \Rightarrow \quad C_1 = \frac{\rho g_x}{\mu} h$$

$$\therefore u(y) = \frac{\rho g_x}{\mu}\left(hy - \frac{y^2}{2}\right)$$

Note:

$\underline{g} = g_x\hat{\imath} + g_y\hat{\jmath}$

where,

$g_x = g\sin\theta$

$g_y = -g\cos\theta$

Analysis:

$$q = \int_0^h u dy = \frac{\rho g_x}{\mu}\frac{h^3}{3}$$

$$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = (\mu)\left(\frac{\rho g_x}{\mu} h\right) = \rho g_x h$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Buckingham Pi Theorem

- For any physically meaningful equation involving $\boldsymbol{n}$ **variables**, such as

$$u_1 = f(u_2, u_3, \cdots, u_n)$$

with minimum number of $\boldsymbol{m}$ **reference dimensions**, the equation can be rearranged into product of $\boldsymbol{r}$ **dimensionless pi terms**.

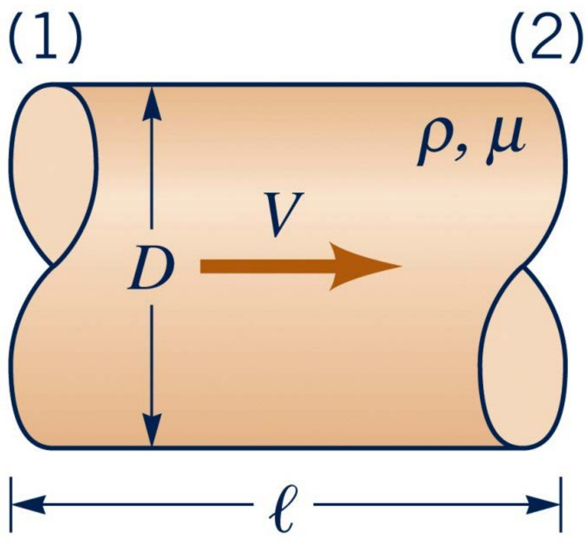
$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_r)$$

where,

$$\boldsymbol{r = n - m}$$

Repeating Variable Method

Example: The pressure drop per unit length Δp_ℓ in a pipe flow is a function of the pipe diameter D and the fluid density ρ, viscosity μ, and velocity V.



$$\Delta p_\ell = (p_1 - p_2)/\ell$$

Repeating Variable Method – Contd.

Step 1: List all variables that are involved in the problem

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Step 2: Express each of the variables in terms of basic dimensions (either MLT or FLT system)

Δp_ℓ	D	ρ	μ	V
$\{ML^{-2}T^{-2}\}$	$\{L\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$	$\{LT^{-1}\}$

Step 3: Determine the required number of pi terms

$$r = n - m = 5 - 3 = 2$$

Step 4: Select m = 3 repeating variables

D (for L), V (for T), and ρ (for M)

Repeating Variable Method – Contd.

Step 5: Form a pi term for one of the non-repeating variables

$$\Pi_1 = D^a V^b \rho^c \Delta p_\ell \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-2}T^{-2}) \doteq M^0 L^0 T^0$$

$$\therefore \Pi_1 = D^{-1} V^{-2} \rho^{-1} \Delta p_\ell = \frac{\Delta p_\ell D}{\rho V^2}$$

Step 6: Repeat step 5 for each of the remaining non-repeating variables

$$\Pi_2 = D^a V^b \rho^c \mu \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1}) \doteq M^0 L^0 T^0$$

$$\therefore \Pi_2 = D^{-1} V^{-1} \rho^{-1} \mu = \frac{\mu}{DV\rho}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Repeating Variable Method – Contd.

Step 7: Check all the resulting pi terms to make sure they are dimensionless and independent

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq F^0 L^0 T^0; \quad \Pi_2 = \frac{\mu}{D V \rho} \doteq F^0 L^0 T^0$$

Step 8: Express the final form as a relationship among the pi terms

$$\Pi_1 = \phi(\Pi_2)$$

or

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Common Dimensionless Parameters for Fluid Flow Problems

Dimensionless Groups	Symbol	Definition	Interpretation
Reynolds number	Re	$\frac{\rho V L}{\mu}$	$\frac{\text{inertia force}}{\text{viscous force}} = \frac{\rho V^2/L}{\mu V/L^2}$
Froude number	Fr	$\frac{V}{\sqrt{gL}}$	$\frac{\text{inertia force}}{\text{gravity force}} = \frac{\rho V^2/L}{\gamma}$
Weber number	We	$\frac{\rho V^2 L}{\sigma}$	$\frac{\text{inertia force}}{\text{surface tension force}} = \frac{\rho V^2/L}{\sigma/L^2}$
Mach number	Ma	$\frac{V}{\sqrt{K/\rho}} = \frac{V}{a}$	$\sqrt{\frac{\text{indertia force}}{\text{compressibility force}}}$
Euler number	C_p	$\frac{\Delta p}{\rho V^2}$	$\frac{\text{pressure force}}{\text{inertia force}} = \frac{\Delta p/L}{\rho V^2/L}$

Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

For,

$$\Pi_1 = \phi(\Pi_2, \ldots, \Pi_n)$$

Similarity requirements:

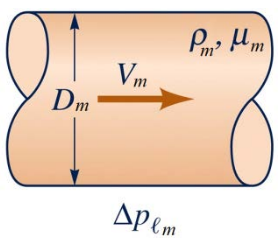
$$\Pi_{2,\text{model}} = \Pi_{2,\text{prototype}}$$
$$\vdots$$
$$\Pi_{n,\text{model}} = \Pi_{n,\text{prototype}}$$

Prediction equation:

$$\Pi_{1,\text{model}} = \Pi_{1,\text{prototype}}$$

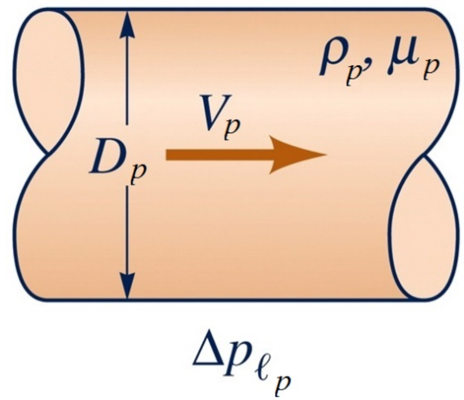
Example (Model Testing)

<u>Model</u>



$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

<u>Prototype</u>



If,

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p} \quad \text{(similarity requirement)}$$

Then,

$$\frac{\Delta p_{\ell_m} D_m}{\rho_m V_m^2} = \frac{\Delta p_{\ell_p} D_p}{\rho_p V_p^2} \quad \text{(Prediction equation)}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Example – Contd.

<u>Model (in water)</u>

- D_m = 0.1 m
- ρ_m = 998 kg/m^3
- μ_m = 1.12 × 10^{-3} N·s/m^2
- V_m = ?
- Δp_{ℓ_m} = 27.6 Pa/m

<u>Prototype (in air)</u>

- D_p = 1 m
- ρ_p = 1.23 kg/m^3
- μ_p = 1.79 × 10^{-5} N·s/m^2
- V_p = 10 m/s
- Δp_{ℓ_m} = ?

Similarity requirement:

$$V_m = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{D_p}{D_m}\right)V_p = \left(\frac{1.23}{998}\right)\left(\frac{1.12\times 10^{-3}}{1.79\times 10^{-5}}\right)\left(\frac{1}{0.1}\right)(10) = \mathbf{7.71\ m/s}$$

Prediction equation:

$$\Delta p_{\ell_p} = \left(\frac{D_m}{D_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2 \Delta p_{\ell_m} = \left(\frac{0.1}{1}\right)\left(\frac{1.23}{998}\right)\left(\frac{10}{7.71}\right)^2 (27.6) = \mathbf{5.72\times 10^{-3}\ Pa/m}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.