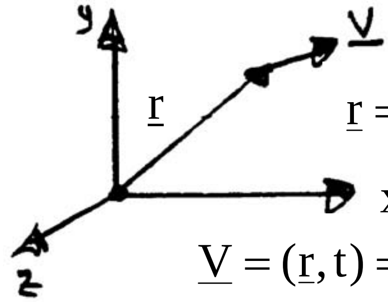
Chapter 4: Fluids Kinematics

4.1 Velocity and Description Methods

Primary dependent variable is fluid velocity vector
$\underline{V} = \underline{V}(\underline{r})$; where $\underline{r}$ is the position vector

If $\underline{V}$ is known then pressure and forces can be determined using techniques to be discussed in subsequent chapters.



$$\underline{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

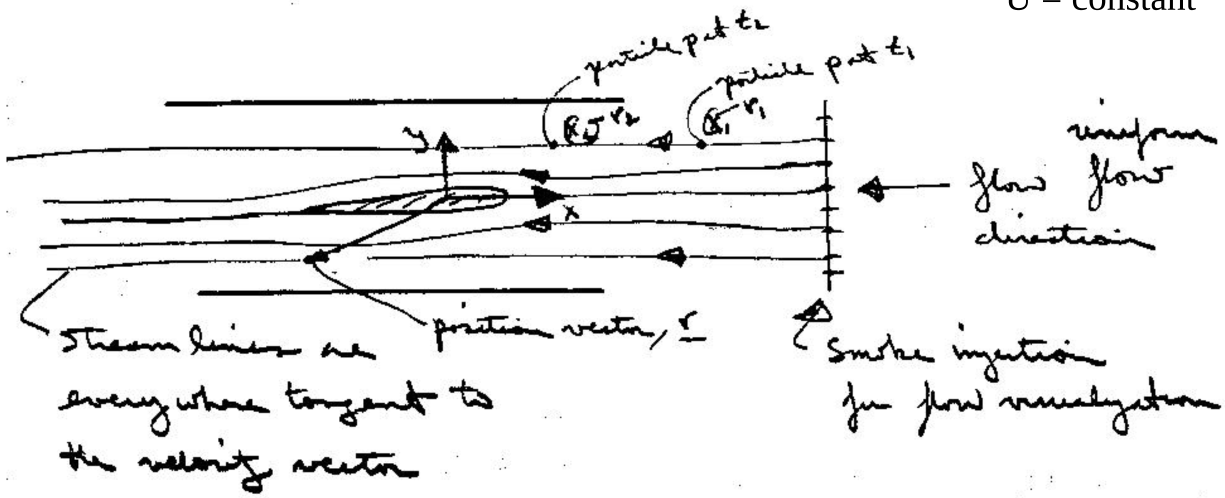
$$\underline{V} = (\underline{r}, t) = u\hat{i} + v\hat{j} + w\hat{k}$$

Consideration of the velocity field alone is referred to as flow field kinematics in distinction from flow field dynamics (force considerations).

Fluid mechanics and especially flow kinematics is a geometric subject and if one has a good understanding of the flow geometry then one knows a great deal about the solution to a fluid mechanics problem.

Consider a simple flow situation, such as an airfoil in a wind tunnel:

U = constant



Velocity: Lagrangian and Eulerian Viewpoints

There are two approaches to analyzing the velocity field: Lagrangian and Eulerian

<u>Lagrangian:</u> keep track of individual fluids particles (i.e., solve $\underline{F} = M\underline{a}$ for each particle)

Say particle p is at position $\underline{r}_1(t_1)$ and at position $\underline{r}_2(t_2)$ then,

$$\underline{V_p} = \lim_{\Delta t \to 0} \frac{\underline{r}_2 - \underline{r}_1}{t_2 - t_1} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$= u_p\hat{i} + v_p\hat{j} + w_p\hat{k}$$

Of course the motion of one particle is insufficient to describe the flow field, so the motion of all particles must be considered simultaneously which would be a very difficult task. Also, spatial gradients are not given directly. Thus, the Lagrangian approach is only used in special circumstances.

<u>Eulerian:</u> focus attention on a fixed point in space

$$\underline{x} = x\hat{i} + y\hat{j} + z\hat{k}$$

In general,

$$\underline{V} = \underline{V}(\underline{x}, t) = \underbrace{u\hat{i} + v\hat{j} + w\hat{k}}_{\text{velocity components}}$$

where,

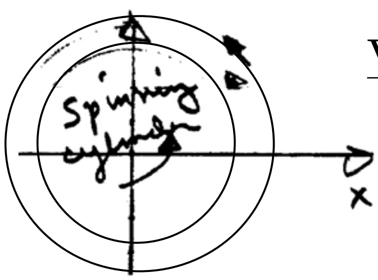
$$u = u(x,y,z,t), \quad v = v(x,y,z,t), \quad w = w(x,y,z,t)$$

This approach is by far the most useful since we are usually interested in the flow field in some region and not the history of individual particles.

However, must transform $\underline{F} = M\underline{a}$ from system to CV (recall Reynolds Transport Theorem (RTT) & CV analysis from thermodynamics)

Ex. Flow around a car

$\underline{V}$ can be expressed in any coordinate system; e.g., polar or spherical coordinates. Recall that such coordinates are called orthogonal curvilinear coordinates. The coordinate system is selected such that it is convenient for describing the problem at hand (boundary geometry or streamlines).



$$\underline{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta$$

$$x = r\cos\theta$$

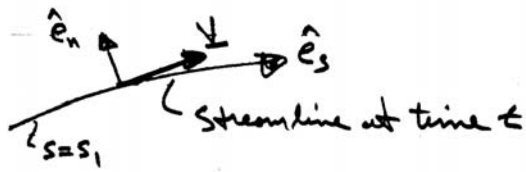
$$y = r\sin\theta$$

$$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Undoubtedly, the most convenient coordinate system is streamline coordinates:

$$\underline{V}(s,t) = v_s(s,t)\hat{e}_s(s,t)$$



However, usually $\underline{V}$ not known a priori and even if known streamlines maybe difficult to generate/determine.

4.2 Flow Visualization and Plots of Fluid Flow Data

See textbook for:

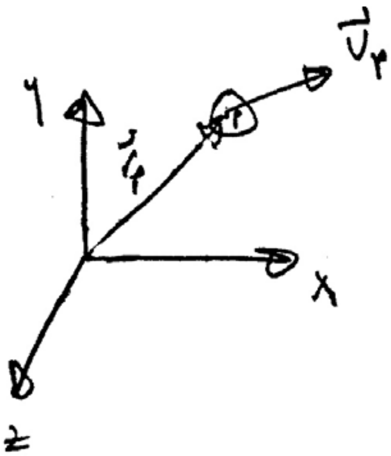
- Streamlines and Streamtubes
- Pathlines
- Streaklines
- Timelines
- Refractive flow visualization techniques
- Surface flow visualization techniques

- Profile plots
- Vector plots
- Contour plots

4.3 Acceleration Field and Material Derivative

The acceleration of a fluid particle is the rate of change of its velocity.

In the Lagrangian approach the velocity of a fluid particle is a function of time only since we have described its motion in terms of its position vector.



$$\underline{r}_p = x_p(t)\hat{i} + y_p(t)\hat{j} + z_p(t)\hat{k}$$

$$\underline{V}_p = \frac{d\vec{r}_p}{dt} = u_p\hat{i} + v_p\hat{j} + w_p\hat{k}$$

$$\underline{a}_p = \frac{d\vec{v}_p}{dt} = \frac{d^2\vec{r}_p}{dt^2} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$a_x = \frac{du_p}{dt} \quad a_y = \frac{dv_p}{dt} \quad a_z = \frac{dw_p}{dt}$$

In the Eulerian approach the velocity is a function of both space and time; consequently,

$$\underline{V} = u(x,y,z,t)\hat{i} + v(x,y,z,t)\hat{j} + w(x,y,z,t)\hat{k}$$

$$\underline{a} = \frac{d\vec{V}}{dt} = \frac{du}{dt}\hat{i} + \frac{dv}{dt}\hat{j} + \frac{dw}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

x,y,z are f(t) since we must follow the particle in evaluating du/dt

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = \underbrace{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}$$

called substantial derivative $\frac{Du}{Dt}$

Similarly for a_y & a_z,

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

In vector notation this can be written concisely

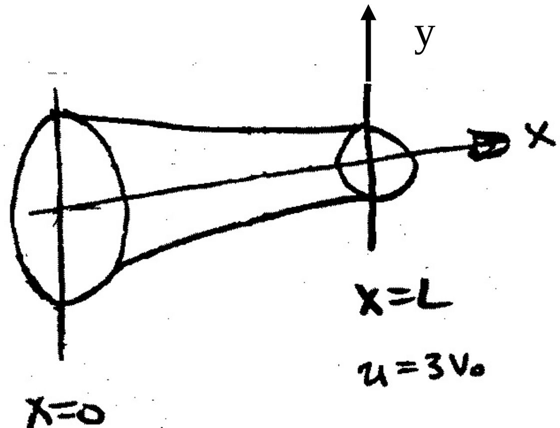
$$\frac{D\underline{V}}{Dt} = \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \quad \text{gradient operator}$$

First term, $\frac{\partial \underline{V}}{\partial t}$, called local or temporal acceleration results from velocity changes with respect to time at a given point. Local acceleration results when the flow is unsteady.

Second term, $\underline{V} \cdot \nabla \underline{V}$, called convective acceleration because it is associated with spatial gradients of velocity in the flow field. Convective acceleration results when the flow is non-uniform, that is, if the velocity changes along a streamline.

The convective acceleration terms are nonlinear which causes mathematical difficulties in flow analysis; also, even in steady flow the convective acceleration can be large if spatial gradients of velocity are large.
Example: Flow through a converging nozzle can be approximated by a one dimensional velocity distribution $u = u(x)$. For the nozzle shown, assume that the velocity varies linearly from $u = V_o$ at the entrance to $u = 3V_o$ at the exit. Compute the acceleration $\frac{D\underline{V}}{Dt}$ as a function of x.



Evaluate $\frac{D\underline{V}}{Dt}$ at the entrance and exit if V_o = 10 ft/s and L =1 ft.

$u(x) = mx + b$
$u(0) = b = V_o$
$m = \frac{\Delta u}{\Delta x} = \frac{3V_o - V_o}{L} = \frac{2V_o}{L}$

We have $\underline{V} = u(x)\hat{i}$, $\frac{Du}{Dt} = u\frac{\partial u}{\partial x} = a_x$

Assume linear variation between inlet and exit

$$u(x) = \frac{2V_o}{L}(x) + V_o = V_o\left(\frac{2x}{L} + 1\right)$$

$$\frac{\partial u}{\partial x} = \frac{2V_0}{L} \Rightarrow \boxed{a_x = \frac{2V_o^2}{L}\left(\frac{2x}{L} + 1\right)}$$

@ x = 0 $\qquad a_x = 200\ \text{ft/s}^2$

@ x = L $\qquad a_x = 600\ \text{ft/s}^2$

4.4 Other Kinematic Descriptions: Types of Motion or Deformation of Fluid Elements

Motion of the fluid element:

[Motion]= [1. Translation] + [2. Rotation] + [3.Volumetric strain]+[4.Shear strain]

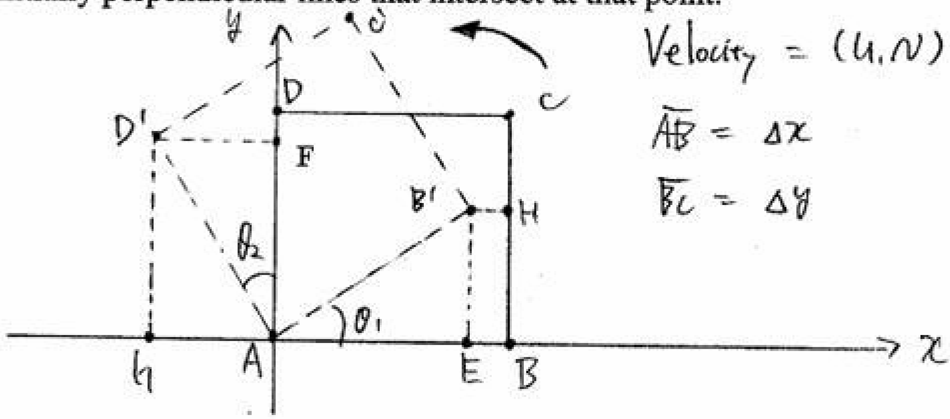
1.Translation:

The rate of translation is described mathematically as the velocity vector

$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

2.Rotation:

The rate of rotation at a point is defined as the average rotation rate of two initially perpendicular lines that intersect at that point.



For example, think about the rotation in x-y plane.

Motion at point B: [Vertical motion]>>[Horizontal motion] because;

$$\overline{EB} = \overline{AB} - \overline{AE} = \Delta x - \Delta x \cos\theta_1$$

$$\overline{EB'} = \overline{BH} = \Delta x \sin\theta_1$$

Since $\theta_1 << 1$,

$$\overline{EB} = \Delta x(1 - \cos\theta_1) \cong 0$$

$$\overline{EB'} = \overline{BH} = \Delta x \sin\theta_1 \cong \Delta x\theta_1$$

Motion at point D: [Horizontal motion]>>[Vertical motion] because;

$$\overline{FD} = \overline{AD} - \overline{AF} = \Delta y - \Delta y \cos\theta_2$$

$$\overline{FD'} = \Delta y \sin\theta_2$$

Since $\theta_2 << 1$,

$$\overline{FD} = \Delta y(1-\cos\theta_2) \cong 0$$

$$\overline{FD'} = \Delta y \sin\theta_2 \cong \Delta y \theta_2$$

Then, think about the deformation derived above in a small time Δt as;

At point B, vertilcal motion dominates $\rightarrow \Delta x \theta_1 = \Delta v \Delta t$

At point D, horizontal motion dominates $\rightarrow \Delta y \theta_2 = -\Delta u \Delta t$

Then, the rate of rotation is taking the average of $\dfrac{\theta_1}{\Delta t}$ and $\dfrac{\theta_2}{\Delta t}$

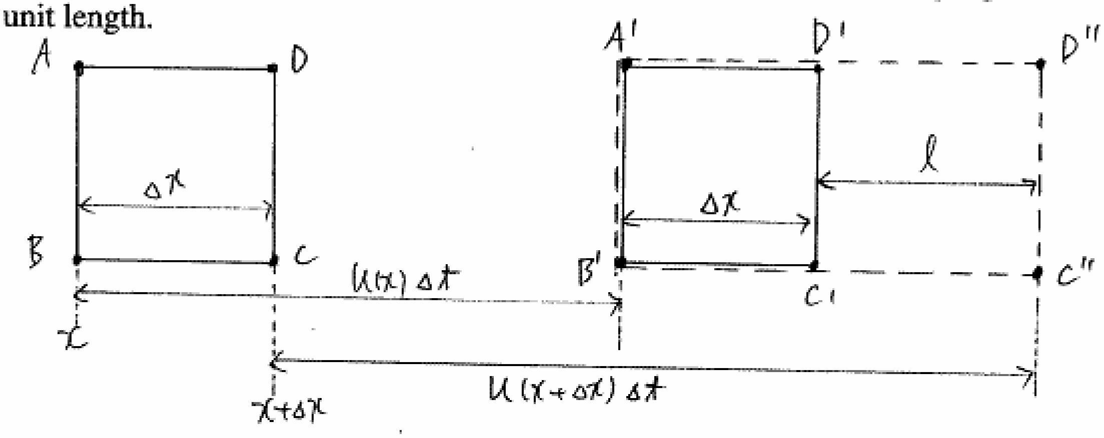
$$\omega = \frac{1}{2}\left(\frac{\theta_1}{\Delta t} + \frac{\theta_2}{\Delta t}\right) = \frac{1}{2}\left(\frac{\Delta v}{\Delta x} - \frac{\Delta u}{\Delta y}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Following the same manner in y-z plane and x-z plane, finally we obtain rate of rotation vector as:

$$\omega = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

3.Volumetric strain

Defined as linear strain. Linear strain rate is the rate of increase in length per unit length.



Stretched length l is

$$l = \overline{BC} + u(x+\Delta x)\Delta t - (\overline{B'C'} + u\Delta t) = u(x+\Delta x)\Delta t - u(x)\Delta t$$

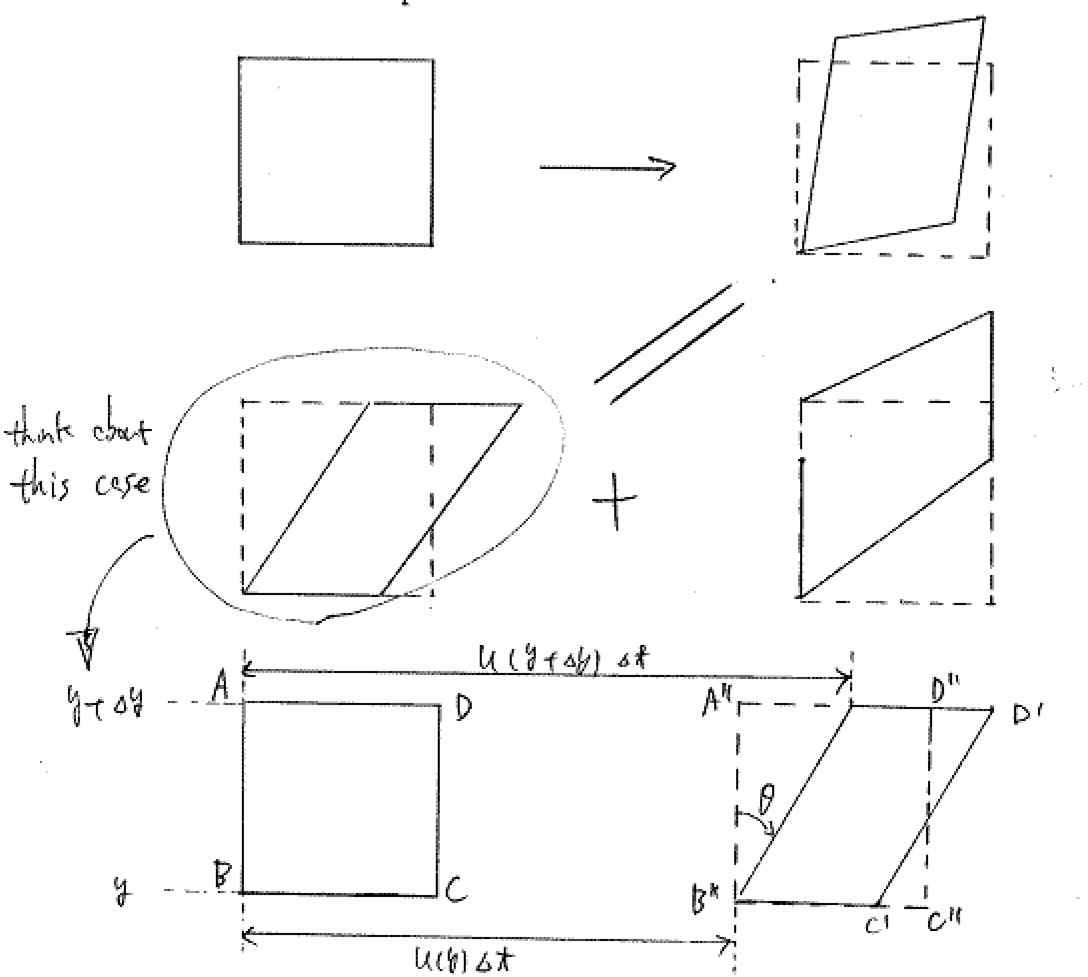
$$\therefore \frac{l}{\Delta t} = u(x+\Delta x) - u(x) \rightarrow \varepsilon = \frac{l}{\Delta t} \cong \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = \frac{\partial u}{\partial x}$$

Following the same manner in y- and z- direction, we obtain the rate of linear strain as;

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \varepsilon_{zz} = \frac{\partial w}{\partial y}$$

4.Shear strain

Shear stress causes shear strain. Shear strain rate at a point is defined as half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point.



First, think about x - direction only

$$\tan\theta_1 = \frac{A''A'}{A''B''}$$

Let $\theta_1 \ll 1$ then

$$\frac{A''A'}{A''B''} = \frac{u(y+\Delta y)\Delta t - u(y)\Delta t}{\Delta y} = \tan\theta_1 \cong \theta_1$$

$$\rightarrow \frac{\theta_1}{\Delta t} = \frac{u(y+\Delta y) - u(y)}{\Delta y} \cong \frac{\partial u}{\partial y}$$

Following the same manner in y-direction (use θ_2) then obtain $\frac{\theta_2}{\Delta t} = \frac{\partial v}{\partial x}$.

Finally, take an average of the two, then $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\theta_1}{\Delta t} + \frac{\theta_2}{\Delta t}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$.

Extend it into three dimensions and finally we obtain the rate of shear strain as;

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right), \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Stretch and Strain are both caused by stress (both normal and parallel to the surface), therefore, those two are combined together as a form called strain rate tensor as follows;

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

Example problem: Deformation rate of fluid element

Consider the steady, two-dimensional velocity field given by

$$\vec{V} = (u, v) = (0.5 + 0.8x)\vec{i} + (1.5 - 0.8y)\vec{j}$$

Calculate the kinematic properties such as;

(a) Rate of translation
(b) Rate of rotation
(c) Rate of linear strain
(d) Rate of shear strain

Solution:

(a) Rate of translation:

$$u = 0.5 + 0.8x, \quad v = 1.5 - 0.8y, \quad w = 0$$

(b) Rate of rotation:

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \frac{1}{2}(0 - 0)\vec{k} = 0$$

(c) Rate of linear strain:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 0.8s^{-1}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -0.8s^{-1}, \quad \varepsilon_{zz} = 0$$

(d) Rate of shear strain:

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}(0 + 0) = 0$$

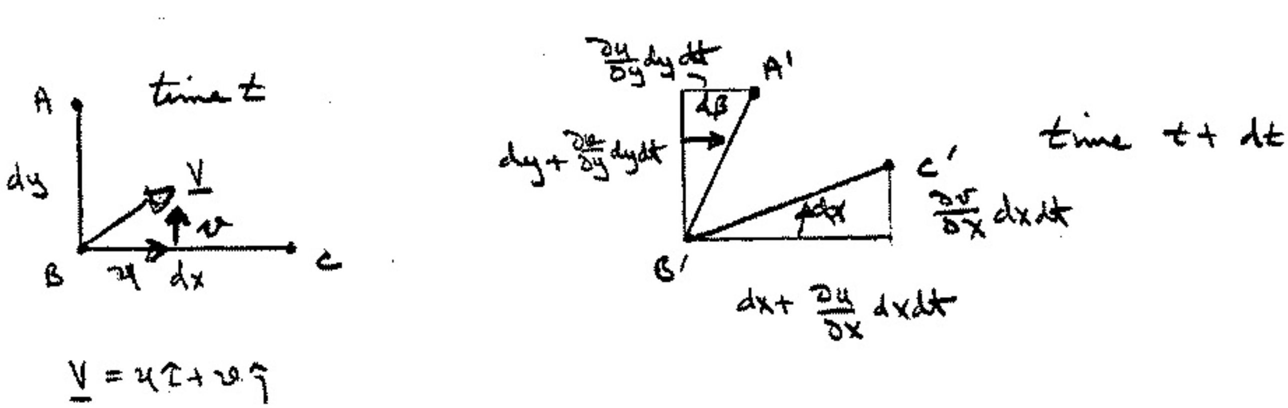
Rotation and Vorticity

$\underline{\Omega}$ = fluid vorticity = 2 × angular velocity = $2\underline{\omega}$

$$= \nabla \times \underline{V} \qquad \text{i.e., curl } \underline{V} \qquad \omega_x = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\hat{k}$$

To show that this definition is correct consider two lines in the fluid



Angular velocity about z axis = average rate of rotation +↺

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right)$$

$$d\alpha = \tan^{-1}\frac{\frac{\partial v}{\partial x}dxdt}{dx + \frac{\partial u}{\partial x}dxdt}$$

$$\lim_{dt\to 0} = \frac{\partial v}{\partial x}dt \qquad \text{i.e.,} \quad \frac{d\alpha}{dt} = \frac{\partial v}{\partial x}$$

$$d\beta = \tan^{-1}\frac{\frac{\partial u}{\partial y}dydt}{dy + \frac{\partial v}{\partial y}dydt}$$

$$\lim_{dt\to 0} = \frac{\partial u}{\partial y}dt \qquad \text{i.e.,} \quad \frac{d\beta}{dt} = \frac{\partial u}{\partial y}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

similarly, $$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

i.e., $\underline{\Omega} = 2\underline{\omega}$

Example problem: Calculation of Vorticity

Consider the following steady, three-dimensional velocity field

$$\vec{V} = (u, v, w) = (3.0 + 2.0x - y)\vec{i} + (2.0x - 2.0y)\vec{j} + \left(0.5xy\right)\vec{k}$$

Calculate the vorticity vector as a function of space $\left(x, y, z\right)$

Solution:

Vorticity vector in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

For $u = 3.0 + 2.0x - y$, $v = 2.0x - 2.0y$, $w = 0.5xy$

$$\vec{\zeta} = \left(0.5x - 0\right)\vec{i} + \left(0 - 0.5y\right)\vec{j} + \left(2.0 - \left(-1\right)\right)\vec{k}$$

$$= \left(0.5x\right)\vec{i} - \left(0.5y\right)\vec{j} + \left(3.0\right)\vec{k}$$