Similarity and Model Testing

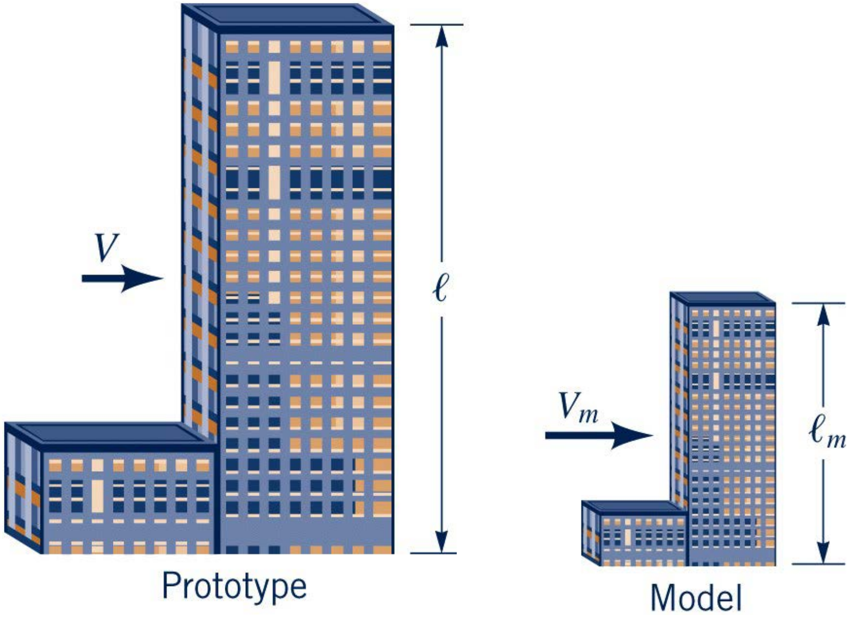
11. 5. 2014

Hyunse Yoon, Ph.D.
Assistant Research Scientist
IIHR-Hydroscience & Engineering
e-mail: hyun-se-yoon@uiowa.edu

Modeling

- **Model:** A representation of a physical system that may be used to predict the behavior of the system in some desired respect

- **Prototype:** The physical system for which the predictions are to be made



Types of Similarity

- Three necessary conditions for complete similarity between a model and a prototype:

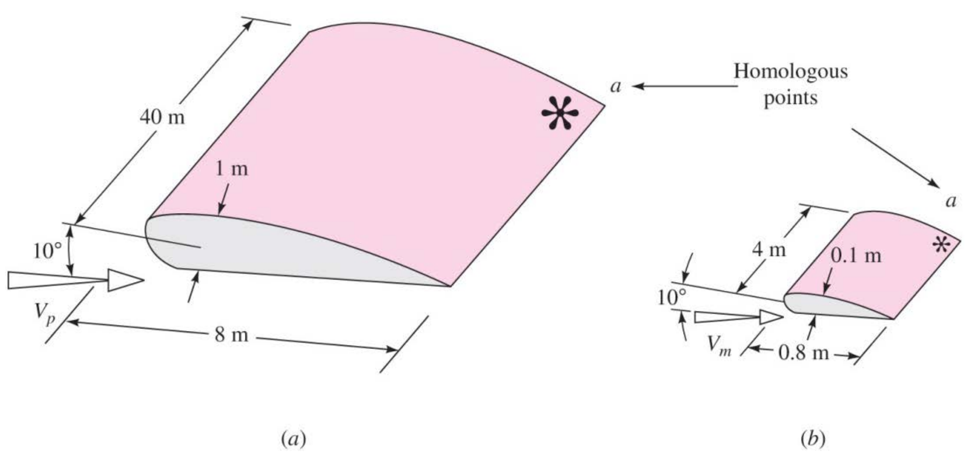
  1) Geometric similarity
  2) Kinematic similarity
  3) Dynamic similarity

1) Geometric Similarity

- A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear scale ratio.

$$\alpha = L_m/L \qquad (\text{or}, \lambda = L/L_m)$$

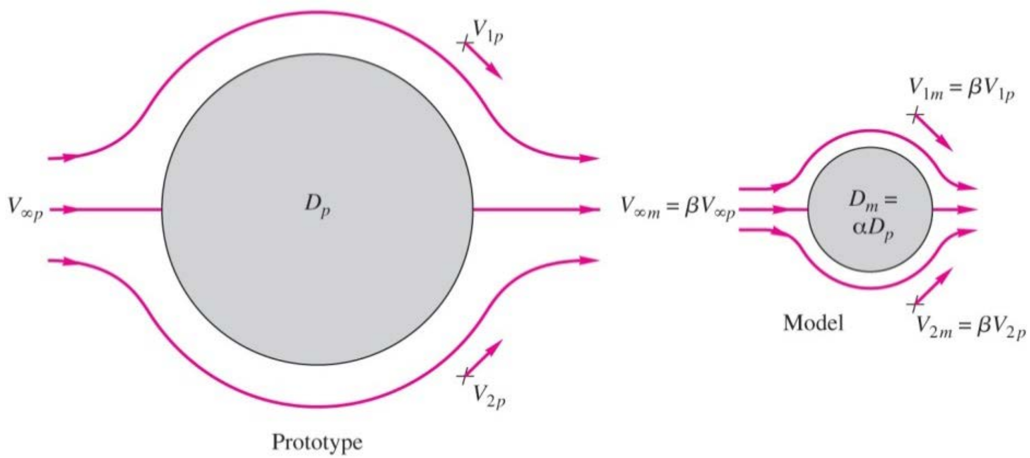
- All angles are preserved in geometric similarity. All flow directions are preserved. The orientation of model and prototype with respect to the surroundings must be identical.



2) Kinematic Similarity

- Kinematic similarity requires that the model and prototype have the same length scale and the same time scale ratio.

- One special case is incompressible frictionless flow with no free surface. These perfect-fluid flows are kinematically similar with independent length and time scales, and no additional parameters are necessary.



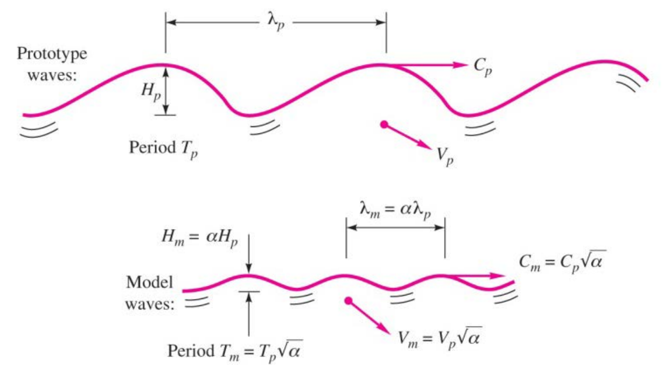
2) Kinematic Similarity – Contd.

- Frictionless flows with a free are kinematically similar if their Fr are equal:

$$Fr_m = \frac{V_m^2}{gL_m} = \frac{V^2}{gL} = Fr$$

or

$$\frac{V_m}{V} = \left(\frac{L_m}{L}\right)^{\frac{1}{2}} = \sqrt{\alpha}$$



- In general, kinematic similarity depends on the achievement of dynamic similarity if viscosity, surface tension, or compressibility is important.

Example 1: Fr Similarity

$\lambda = \frac{1}{25}$

8.52 If the scale ratio between a model spillway and its prototype is $\frac{1}{25}$, what velocity and discharge ratio will prevail between model and prototype? If the prototype discharge is 3000 m^3/s, what is the model discharge?

Fig. 5.9 Hydraulic model of the Bluestone Lake Dam on the New River near Hinton, West Virginia. The model scale is 1:65 both vertically and horizontally, and the Reynolds number, though far below the prototype value, is set high enough for the flow to be turbulent. *(Courtesy of the U.S. Army Corps of Engineers Waterways Experiment Station.)*

Example 1: Fr Similarity – Contd.

For Fr similarity,

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V}{\sqrt{gL}} \qquad \Rightarrow \qquad \frac{V_m}{V} = \sqrt{\frac{L_m}{L}} = \alpha^{\frac{1}{2}} = \left(\frac{1}{25}\right)^{\frac{1}{2}} = \frac{\mathbf{1}}{\mathbf{5}}$$

Since $Q = VA$,

$$\frac{Q_m}{Q} = \frac{V_m A_m}{VA} = \left(\frac{V_m}{V}\right)\left(\frac{A_m}{A}\right) = \alpha^{\frac{1}{2}}\left(\frac{A_m}{A}\right)$$

Also,

$$\frac{A_m}{A} = \left(\frac{L_m}{L}\right)^2 = \alpha^2$$

Thus,

$$\frac{Q_m}{Q} = \alpha^{\frac{1}{2}}\alpha^2 = \alpha^{\frac{5}{2}} = \left(\frac{1}{25}\right)^{\frac{5}{2}} = \frac{\mathbf{1}}{\mathbf{3,125}}$$

$$\therefore Q_m = \alpha^{\frac{5}{2}} Q = \left(\frac{1}{25}\right)^{\frac{5}{2}} (3000) = \mathbf{0.96\ m^3/s}$$

3) Dynamic Similarity

- Dynamic similarity exists when the model and the prototype have the same length scale ratio (i.e., geometric similarity), time scale ratio (i.e., kinematic similarity), and force scale (or mass scale) ratio.

- To be ensure of identical force and pressure coefficients between model and prototype:
    1. Compressible flow: Re and Ma are equal
    2. Incompressible flow:
        a. With no free surface: Re are equal
        b. With a free surface: Re and Fr are equal
        c. If necessary, We and Ca are equal

Theory of Models

- Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and then prototype.

- For prototype:

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \dots, \Pi_n)$$

- For model:

$$\Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \dots, \Pi_{nm})$$

- Similarity requirement*

$$\Pi_{2m} = \Pi_2$$
$$\Pi_{3m} = \Pi_3$$
$$\vdots$$
$$\Pi_{nm} = \Pi_n$$

- Prediction equation

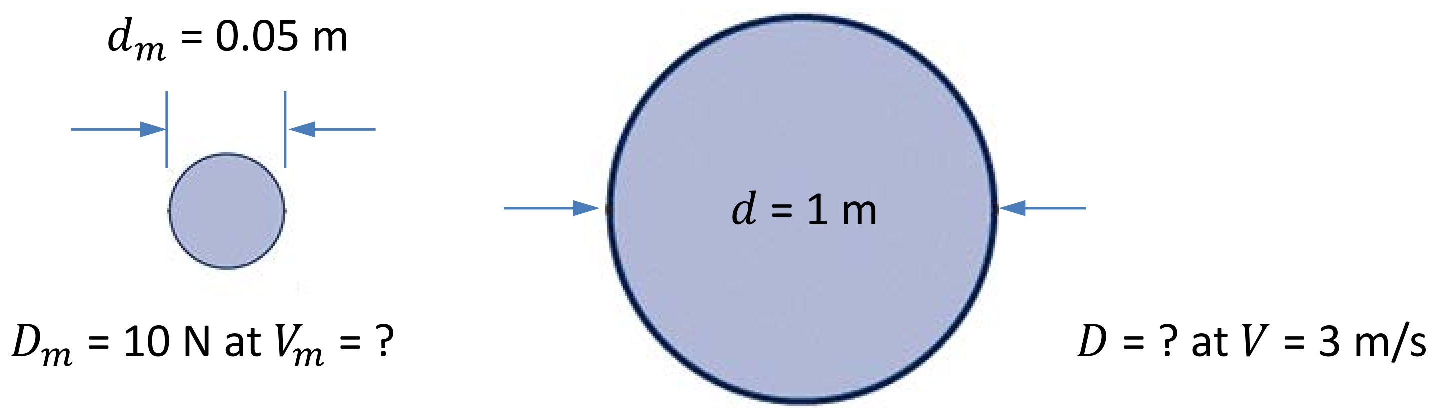
$$\Pi_1 = \Pi_{1m}$$

*Also referred as the model design conditions or modeling laws.

Example 2: Re Similarity

Drag measurements were taken for a 5-cm diameter sphere in water at 20°C to predict the drag force of a 1-m diameter balloon rising in air with standard temperature and pressure. Determine (a) the sphere velocity if the balloon was rising at 3 m/s and (b) the drag force of the balloon if the resulting sphere drag was 10 N. Assume the drag D is a function of the diameter d, the velocity V, and the fluid density ρ and kinematic viscosity ν.

$$D = f(d, V, \rho, \nu)$$



water
model

air
prototype

Example 2: Re Similarity – Contd.

- Dimensional analysis

$$\frac{D}{\rho V^2 d^2} = \phi\left(\frac{Vd}{\nu}\right)$$

- Similarity requirement

$$\frac{Vd}{\nu} = \frac{V_m d_m}{\nu_m}$$

- Prediction equation

$$\frac{D}{\rho V^2 d^2} = \frac{D_m}{\rho_m V_m^2 d_m^2}$$

Example 2: Re Similarity – Contd.

(a) From the similarity requirement:

$$\frac{Vd}{\nu} = \frac{V_m d_m}{\nu_m}$$

or

$$V_m = \left(\frac{\nu_m}{\nu}\right)\left(\frac{d}{d_m}\right)V$$

$$= \left(\frac{1.004 \times 10^{-6}\ \mathrm{m^2/s}}{1.45 \times 10^{-5}\ \mathrm{m^2/s}}\right)\left(\frac{1\ \mathrm{m}}{0.05\ \mathrm{m}}\right)(3\ \mathrm{m/s})$$

$$\therefore V = 4.15\ \mathrm{m/s}$$

Example 2: Re Similarity – Contd.

(b) From the prediction equation:

$$\frac{D}{\rho V^2 d^2} = \frac{D_m}{\rho_m V_m^2 d_m^2}$$

or

$$D = \left(\frac{\rho}{\rho_m}\right)\left(\frac{V}{V_m}\right)^2\left(\frac{d}{d_m}\right)^2 D_m$$

$$= \left(\frac{1.23\,\mathrm{kg/m^3}}{998\,\mathrm{kg/m^3}}\right)\left(\frac{3\ \mathrm{m/s}}{4.15\ \mathrm{m}/s}\right)^2\left(\frac{1\ \mathrm{m}}{0.05\ \mathrm{m}}\right)^2 (10\ \mathrm{N})$$

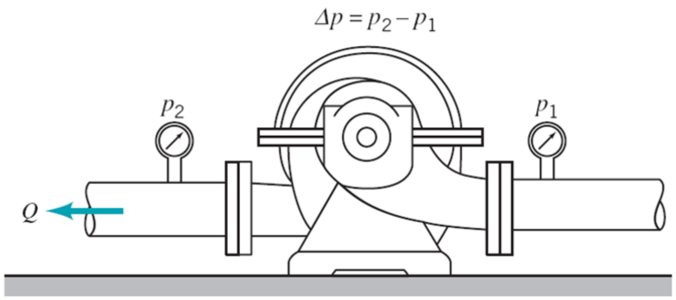
$$\therefore D = 2.6\ \mathrm{N}$$

Example 3

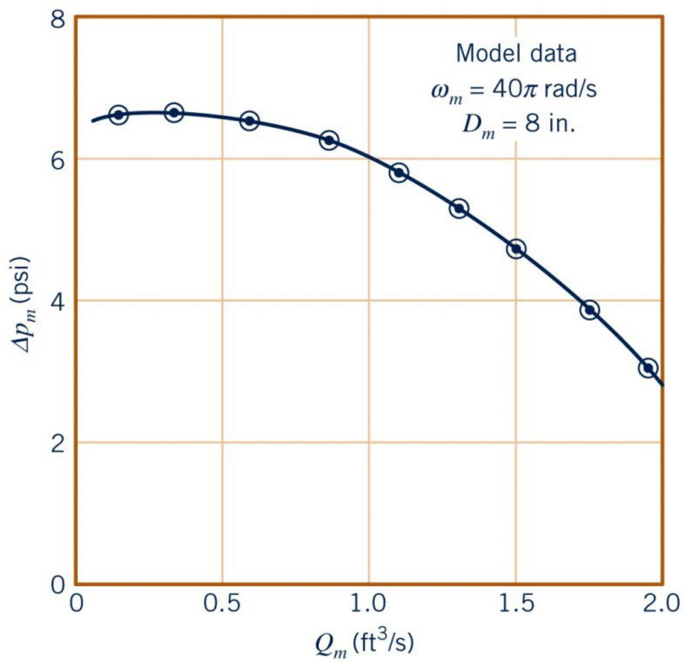
7.79 The pressure rise, Δp, across a centrifugal pump of a given shape can be expressed as

$$\Delta p = f(D, \omega, \rho, Q)$$

where D is the impeller diameter, ω the angular velocity of the impeller, ρ the fluid density, and Q the volume rate of flow through the pump. A model pump having a diameter of 8 in. is tested in the laboratory using water. When operated at an angular velocity of 40π rad/s the model pressure rise as a function of Q is shown in Fig. P7.79. Use this curve to predict the pressure rise across a geometrically similar pump (prototype) for a prototype flowrate of 6 ft³/s. The prototype has a diameter of 12 in. and operates at an angular velocity of 60π rad/s. The prototype fluid is also water.



Centrifugal pump



Property	Model	Prototype
Diameter, D	8 in.	12 in.
Angular velocity, ω	40π rad/s	60π rad/s
Flow rate, Q	?	6 ft^3/s
Fluid	water	water

Example 3 – Contd.

- Dimensional analysis

Δp	D	ω	ρ	Q
$\{ML^{-1}T^{-2}\}$	$\{L\}$	$\{T^{-1}\}$	$\{ML^{-3}\}$	$\{L^3T^{-1}\}$

$$r = n - m = 5 - 3 = 2$$

m = 3 repeating variables: D for L, ω for T, and ρ for M

$$\Pi_1 = D^a\omega^b\rho^c\Delta p \doteq (L)^a(T^{-1})^b(ML^{-3})^c(ML^{-1}T^{-2})$$

$$\doteq L^{(a-3c-1)}T^{(-b-2)}M^{(c+1)} \doteq L^0T^0M^0$$

$$\therefore \Pi_1 = D^{-2}\,\omega^{-2}\rho^{-1}\Delta p = \frac{\Delta p}{\rho\omega^2D^2}$$

Example 3 – Contd.

- Dimensional analysis – contd.

$$\Pi_2 = D^a \omega^b \rho^c Q \doteq (L)^a (T^{-1})^b (ML^{-3})^c (L^3 T^{-1})$$

$$\doteq L^{(a-3c+3)} T^{(-b-1)} M^c \doteq L^0 T^0 M^0$$

$$\therefore \Pi_2 = D^{-3}\, \omega^{-1} Q = \frac{Q}{\omega D^3}$$

- Dimensionless eq.

$$\frac{\Delta p}{\rho \omega^2 D^2} = \phi\left(\frac{Q}{\omega D^3}\right)$$

Example 3 – Contd.

- Similarity requirement

$$\frac{Q}{\omega D^3} = \frac{Q_m}{\omega_m D_m^3}$$

or

$$Q_m = \left(\frac{\omega_m}{\omega}\right)\left(\frac{D_m}{D}\right)^3 Q$$

$$= \left(\frac{40\pi \ \text{rad/s}}{60\pi \ \text{rad/s}}\right)\left(\frac{8 \text{ in.}}{12 \text{ in.}}\right)^3 (6 \ \text{ft}^3/\text{s})$$

$$\therefore Q_m = 1.19 \, \text{ft}^3/\text{s}$$

Example 3 – Contd.

- Prediction equation

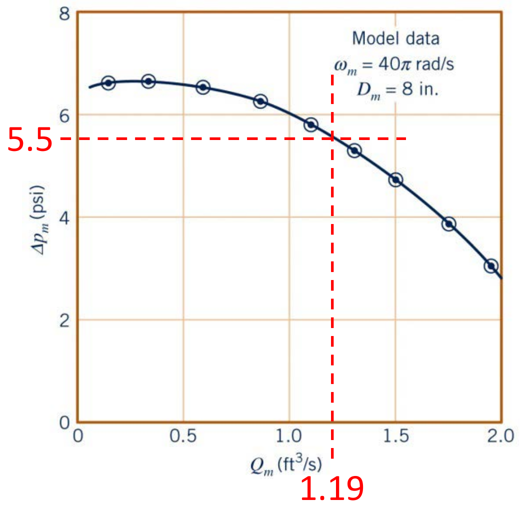
$$\frac{\Delta p}{\rho \omega^2 D^2} = \frac{\Delta p_m}{\rho_m \omega_m^2 D_m^2}$$

or

$$\Delta p = \left(\frac{\rho}{\rho_m}\right)\left(\frac{\omega}{\omega_m}\right)^2 \left(\frac{D}{D_m}\right)^2 \Delta p_m$$

$$= (1)\left(\frac{60\pi \text{ rad/s}}{40\pi \text{ rad/s}}\right)^2 \left(\frac{12 \text{ in.}}{8 \text{ in.}}\right)^2 (5.5\, psi)$$

$$\therefore \Delta p = 27.8\, psi$$



Distorted Models

- It is not always possible to satisfy all the known similarity requirements.

- Models for which one or more similarity requirements are not satisfied are called “distorted models.”

Model Testing in Water

(with a free surface)

- Geometric similarity

$$\frac{L_m}{L} = \alpha$$

- Fr similarity

$$\frac{V_m}{\sqrt{gL_m}} = \frac{V}{\sqrt{gL}} \qquad \Rightarrow \qquad \frac{V_m}{V} = \left(\frac{L_m}{L}\right)^{1/2} = \sqrt{\alpha}$$

- Re similarity

$$\frac{V_m L_m}{\nu_m} = \frac{VL}{\nu} \qquad \Rightarrow \qquad \frac{\nu_m}{\nu} = \frac{L_m}{L}\frac{V_m}{V} = \alpha\sqrt{\alpha} = \alpha^{\frac{3}{2}}$$

Example 4

EXAMPLE 7–11 **Model Lock and River**

In the late 1990s the U.S. Army Corps of Engineers designed an experiment to model the flow of the Tennessee River downstream of the Kentucky Lock and Dam (Fig. 7–43). Because of laboratory space restrictions, they built a scale model with a length scale factor of $L_m/L_p = 1/100$. Suggest a liquid that would be appropriate for the experiment.

FIGURE 7–43
A 1:100 scale model constructed to investigate navigation conditions in the lower lock approach for a distance of 2 mi downstream of the dam. The model includes a scaled version of the spillway, powerhouse, and existing lock. In addition to navigation, the model was used to evaluate environmental issues associated with the new lock and required railroad and highway bridge relocations. The view here is looking upstream toward the lock and dam. At this scale, 52.8 ft on the model represents 1 mi on the prototype. A pickup truck in the background gives you a feel for the model scale.

Photo courtesy of the U.S. Army Corps of Engineers, Nashville.

Example 4 – Contd.

- For water at atmospheric pressure and at T = 15.6°C, the prototype kinematic viscosity is ν = 1.12 × 10^{-6} m^2/s.
- Required kinematic viscosity of model liquid:

$$\nu_m = \nu\alpha^{\frac{3}{2}} = (1.12 \times 10^{-6})\left(\frac{1}{100}\right)^{\frac{3}{2}} = 1.12 \times 10^{-9}\ \mathrm{m}^2/s$$

- Even liquid mercury has a kinematic viscosity of order 10^{-7} m^2/s – still two orders of magnitude too large to satisfy the dynamic similarity. In addition, it would be too expensive and hazardous to use in this model test.

Liquid	Temperature (°C)	Density, ρ (kg/m^3)	Specific Weight, γ (kN/m^3)	Dynamic Viscosity, μ ($N \cdot s/m^2$)	Kinematic Viscosity, ν (m^2/s)	Surface Tension,[a] σ (N/m)	Vapor Pressure, p_v [N/m^2 (abs)]	Bulk Modulus,[b] E_v (N/m^2)
Carbon tetrachloride	20	1,590	15.6	9.58 E − 4	6.03 E − 7	2.69 E − 2	1.3 E + 4	1.31 E + 9
Ethyl alcohol	20	789	7.74	1.19 E − 3	1.51 E − 6	2.28 E − 2	5.9 E + 3	1.06 E + 9
Gasoline[c]	15.6	680	6.67	3.1 E − 4	4.6 E − 7	2.2 E − 2	5.5 E + 4	1.3 E + 9
Glycerin	20	1,260	12.4	1.50 E + 0	1.19 E − 3	6.33 E − 2	1.4 E − 2	4.52 E + 9
Mercury	20	13,600	133	1.57 E − 3	1.15 E − 7	4.66 E − 1	1.6 E − 1	2.85 E + 10
SAE 30 oil[c]	15.6	912	8.95	3.8 E − 1	4.2 E − 4	3.6 E − 2	—	1.5 E + 9
Seawater	15.6	1,030	10.1	1.20 E − 3	1.17 E − 6	7.34 E − 2	1.77 E + 3	2.34 E + 9
Water	15.6	999	9.80	1.12 E − 3	1.12 E − 6	7.34 E − 2	1.77 E + 3	2.15 E + 9

Model Testing in Water – Contd.

(with a free surface)

Alternatively, by keeping $\nu_m = \nu$,

- Re similarity

$$\frac{V_m L_m}{\nu_m} = \frac{VL}{\nu} \qquad \Rightarrow \qquad \frac{V_m}{V} = \frac{L}{L_m} = \alpha^{-1}$$

- Fr similarity

$$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V}{\sqrt{gL}} \qquad \Rightarrow \qquad \frac{g_m}{g} = \left(\frac{V_m}{V}\right)^2 \left(\frac{L}{L_m}\right) = \alpha^{-2}\alpha^{-1} = \alpha^{-3}$$

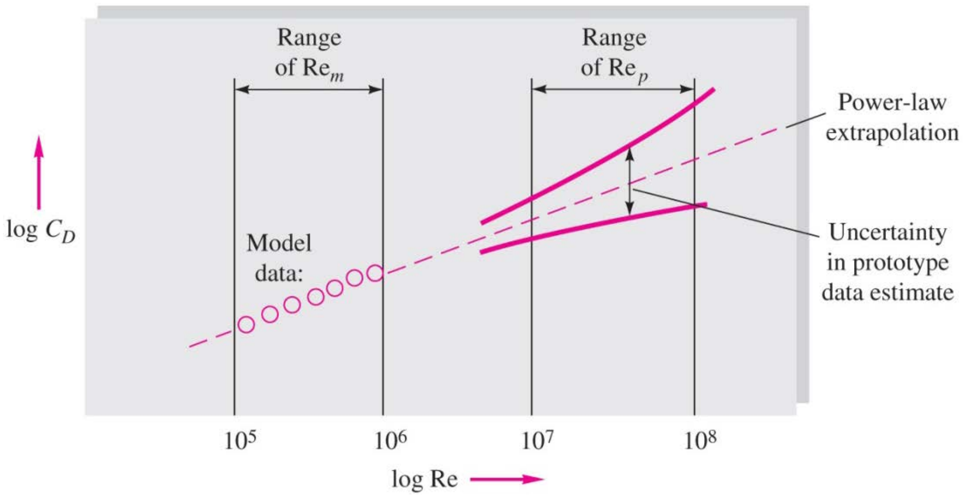
For example, for α = 1/100,

$$\frac{g_m}{g} = \left(\frac{1}{100}\right)^{-3} = 1 \times 10^6$$

$\Rightarrow$ Again, impossible to achieve.

Model Testing in Water – Contd.
(with a free surface)

- In practice, water is used for both the model and the prototype, and the Re similarity is unavoidably violated.
- The low-Re model data are used to estimate by extrapolation the desired high-Re prototype data.
- There is considerable uncertainty in using extrapolation, but no other practical alternative in model testing.



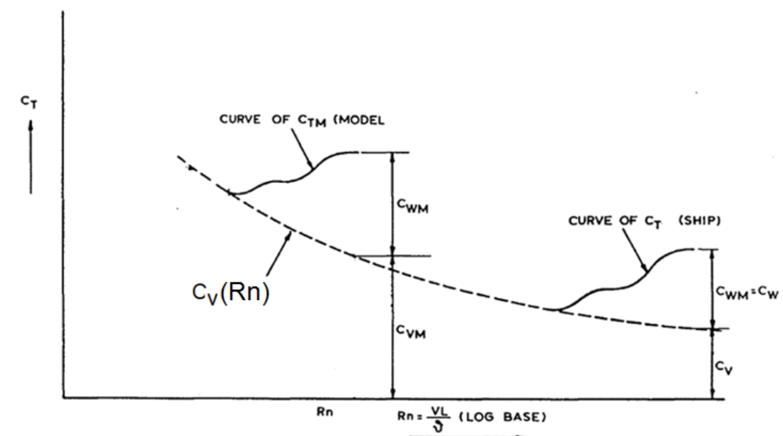
Model Testing in Water – Contd.

(with a free surface: Ship model testing)

- Assume:

$$C_T = f(Re, Fr) = C_w(Fr) + C_v(Re)$$

  - C_T = Total resistance coefficient
  - C_W = Wave resistance coefficient
  - C_v = Viscous friction resistance coefficient



- Model Testing with the Fr similarity

$$C_w = C_{wm} = C_{Tm} - C_{vm}$$

- Extrapolation (with $C_{wm} = Cw$)

$$C_T = C_w + C_v = (C_{Tm} - C_{vm}) + C_v$$

- $C_{vm}(Re_m)$ and $C_v(Re)$ obtained from flat plate data with same surface area of the model and the proto type, respectively.

Model Testing in Air

- Geometric similarity:

$$\frac{L_m}{L} = \alpha$$

- Ma similarity:

$$\frac{V_m}{a_m} = \frac{V}{a} \qquad \Rightarrow \qquad \frac{V_m}{V} = \frac{a_m}{a}$$

- Re similarity:

$$\frac{L_m V_m}{\nu_m} = \frac{LV}{\nu} \qquad \Rightarrow \qquad \frac{\nu_m}{\nu} = \frac{L_m}{L}\frac{V_m}{V} = \alpha \cdot \frac{a_m}{a}$$

- Since the prototype is an air operation, need a model fluid of low viscosity and high speed of sound, e.g., hydrogen – too expensive and danger.
- Re similarity is commonly violated in wind tunnel testing.

Example 5

7.46 To test the aerodynamics of a new prototype automobile, a scale model will be tested in a wind tunnel. For dynamic similarity, it will be required to match Reynolds number between model and prototype. Assuming that you will be testing a one-tenth-scale model and both model and prototype will be exposed to standard air pressure, will it be better for the wind tunnel air to be colder or hotter than standard sea-level air temperature of 15 °C? Why?

Example 5 – Contd.

- Re similarity:

$$\frac{V_m L_m}{\nu_m} = \frac{VL}{\nu}$$

or

$$V_m = \left(\frac{\nu_m}{\nu}\right)\left(\frac{L}{L_m}\right)V = 10\left(\frac{\nu_m}{\nu}\right)V$$

since L_m/L = 1/10. If $\nu_m/\nu = 1$,

$$V_m = 10V$$

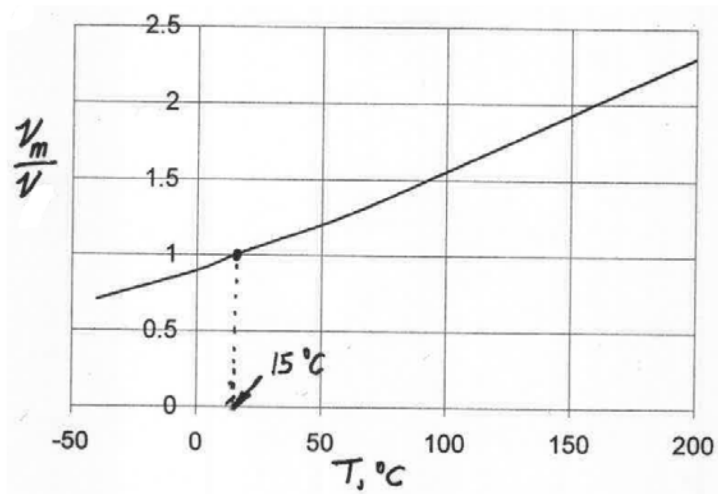
- Also, if V = 55 mph,

$$V_m = 550 \text{ mph}$$

$\Rightarrow$ Too large for simple tests. In addition, Ma = 0.7 for the model and the compressibility effects become important while they are not for the prototype with Ma = 0.07. We will need $\nu_m/\nu < 1$ for more realistic V_m.

Example 5 – Contd.

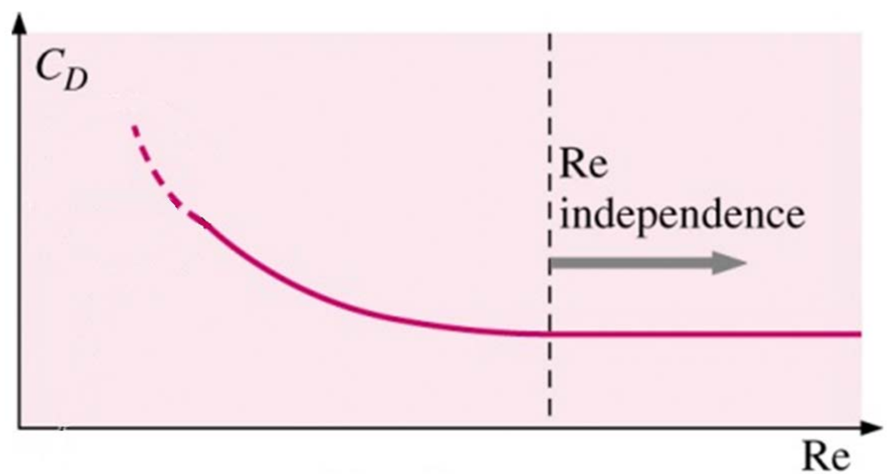
- ν variation with T for air at standard atmospheric pressure (scaled with ν at 10°C):



- Thus, it would be better to have a colder wind tunnel. However, even with T = -40°C, which gives $\nu_m/\nu = 0.707$, the required V_m = 10(0.707)(55) = 389 mph.

Model Testing in Air – Contd.

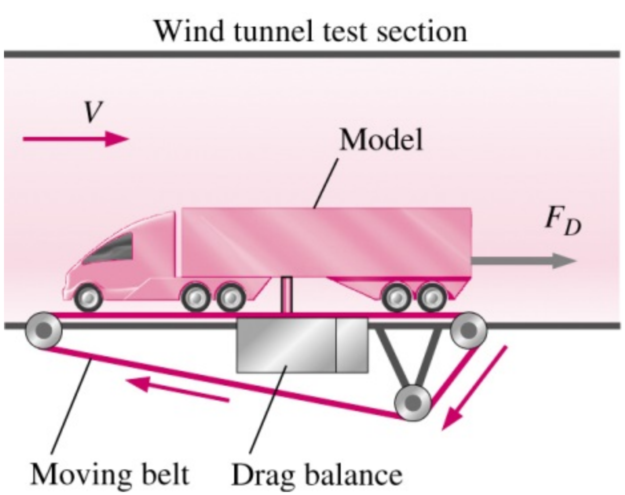
- In practice, wind tunnel tests are performed at several speeds near the maximum operating speed, and then extrapolate the results to the full-scale Reynolds number.
- While drag coefficient C_D is a strong function of Reynolds number at low values of Re, for flows over many objects (especially "bluff" objects), the flow is **Reynolds number independent** above some threshold value of Re.



Example 6

EXAMPLE 7–10 **Model Truck Wind Tunnel Measurements**

A one-sixteenth scale model tractor-trailer truck (18-wheeler) is tested in a wind tunnel as sketched in Fig. 7–38. The model truck is 0.991 m long, 0.257 m tall, and 0.159 m wide. During the tests, the moving ground belt speed is adjusted so as to always match the speed of the air moving through the test section. Aerodynamic drag force F_D is measured as a function of wind tunnel speed; the experimental results are listed in Table 7–7. Plot the drag coefficient C_D as a function of the Reynolds number Re, where the area used for the calculation of C_D is the frontal area of the model truck (the area you see when you look at the model from upstream), and the length scale used for calculation of Re is truck width W. Have we achieved dynamic similarity? Have we achieved Reynolds number independence in our wind tunnel test? Estimate the aerodynamic drag force on the prototype truck traveling on the highway at 26.8 m/s. Assume that both the wind tunnel air and the air flowing over the prototype car are at 25°C and standard atmospheric pressure.



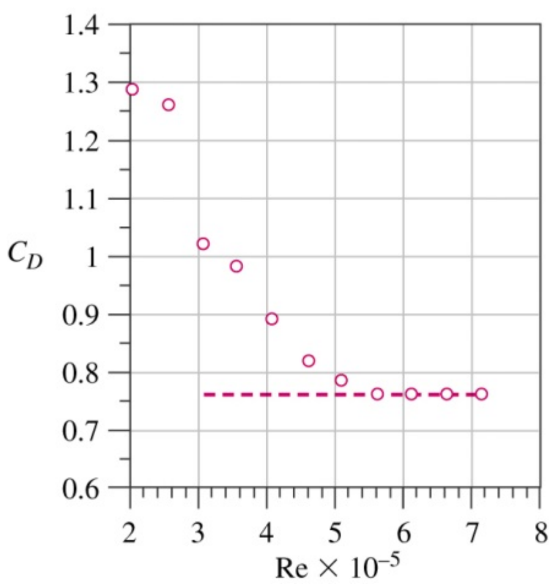
$$\frac{F_D}{\frac{1}{2}\rho V^2 A} = \phi\left(\frac{\rho V W}{\mu}\right)$$

- W = Width of the truck
- A = Frontal area

Example 6 – Contd.

Wind tunnel data: aerodynamic drag force on a model truck as a function of wind tunnel speed

V, m/s	F_D, N
20	12.4
25	19.0
30	22.1
35	29.0
40	34.3
45	39.9
50	47.2
55	55.5
60	66.0
65	77.6
70	89.9



$$(Re_m)_{max} = \frac{\rho_m V_m W_m}{\mu_m} = \frac{\left(1.184\frac{\text{kg}}{\text{m}^3}\right)\left(26.8\frac{\text{m}}{\text{s}}\right)(0.159\text{ m})}{1.849\times10^{-5}\text{ kg}\cdot\text{s/m}} = 7.13\times10^5$$

$$Re = \frac{\rho V W}{\mu} = \frac{\left(1.184\frac{\text{kg}}{\text{m}^3}\right)\left(26.8\frac{\text{m}}{\text{s}}\right)(16\times0.159\text{ m})}{1.849\times10^{-5}\text{ kg}\cdot\text{s/m}} = 4.37\times10^6$$

Reynolds number independence is achieved for $Re > 5.5\times10^5$, thus

$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} = C_{Dm} = 0.76$ from the wind tunnel data

$$\therefore F_D = \frac{1}{2}\rho V^2 A\cdot C_D$$

$$= \frac{1}{2}\left(1.184\frac{\text{kg}}{\text{m}^3}\right)\left(26.8\frac{\text{m}}{\text{s}}\right)^2(16^2\times0.159\text{m}\times0.257\text{m})(0.76)$$

$$= 3.4\ k\text{N}$$