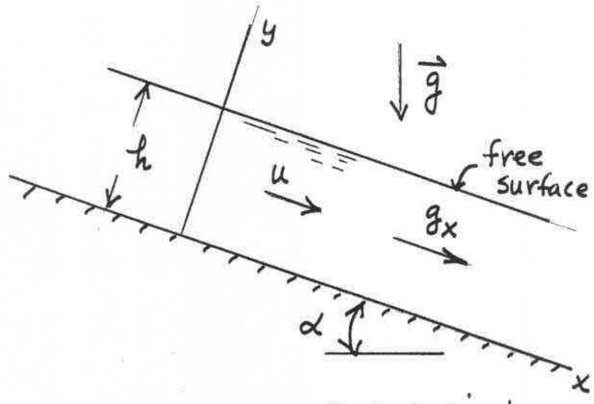
6.93

6.93 A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.



$g_x = g \sin\alpha$

With the coordinate system shown in the figure $v=0$, $w=0$, and from the continuity equation $\frac{\partial u}{\partial x}=0$. Thus, from the x-component of the Navier-Stokes equations (Eq. 6.127a),

$$0 = -\frac{\partial p}{\partial x} + \rho g \sin\alpha + \mu \frac{d^2u}{dy^2} \qquad (1)$$

Also, since there is a free surface, there cannot be a pressure gradient in the x-direction so that $\frac{\partial p}{\partial x}=0$ and Eq. (1) can be written as

$$\frac{d^2u}{dy^2} = -\frac{\rho g}{\mu}\sin\alpha$$

Integration yields

$$\frac{du}{dy} = -\left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_1 \qquad (2)$$

Since the shearing stress

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

equals zero at the free surface ($y=h$) it follows that

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = h$$

so that the constant in Eq. (2) is

$$C_1 = \frac{\rho g}{\mu}\sin\alpha$$

Integration of Eq. (2) yields

$$u = -\left(\frac{\rho g}{\mu}\sin\alpha\right)\frac{y^2}{2} + \left(\frac{\rho g}{\mu}\sin\alpha\right)y + C_2$$

Since $u=0$ at $y=0$, it follows that $C_2=0$, and therefore

$$u = \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)$$

The flowrate per unit width can be expressed as $q = \int_0^h u\,dy$ so that

$$q = \int_0^h \frac{\rho g}{\mu}\sin\alpha\left(hy - \frac{y^2}{2}\right)dy = \underline{\underline{\frac{\rho g h^3 \sin\alpha}{3\mu}}}$$