RTT for Moving Control Volumes

10. 18. 2013

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RTT for moving and deforming CV

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt}\int_{CV} \beta\rho dV + \int_{CS} \beta\rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

where, $\underline{V_R} = \underline{V} - \underline{V_S}$

Non-deforming (but moving) CV and if steady flow:

$$\frac{DB_{sys}}{Dt} = \int_{CV} \frac{\partial}{\partial t}(\beta\rho)d\forall + \int_{CS} \beta\rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

$$\therefore \frac{DB_{sys}}{Dt} = \int_{CS} \beta\rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

Conservation of Mass

$B = M$ and $\beta = 1$:

$$\frac{DM_{sys}}{Dt} = 0 = \int_{CS} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

$$\therefore \int_{CS} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA = 0$$

For pipe flows:

$$\rho_1 V_{1R} A_1 = \rho V_{2R} A_2$$

Conservation of Momentum

$B = M\underline{V}$ and $\beta = \underline{V}$:

$$\frac{D\left(M\underline{V}\right)_{sys}}{Dt} = \sum \underline{F} = \int_{CS} \underline{V} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

However, $\underline{V} = \underline{V_R} + \underline{V_S}$,

$$\sum \underline{F} = \int_{CS} \left(\underline{V_R} + \underline{V_S}\right) \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

$$= \int_{CS} \underline{V_R} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA + \int_{CS} \underline{V_S} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

Note: $\underline{V_R} = \underline{V} - \underline{V_S}$

Conservation of Momentum-Contd.

If $\underline{V_S}$ = constant,

$$\sum \underline{F} = \int_{CS} \underline{V_R} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA + \underline{V_S} \underbrace{\int_{CS} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA}_{\substack{=0 \\ \text{Conservation} \\ \text{of mass}}}$$

$$\therefore \sum \underline{F} = \int_{CS} \underline{V_R} \rho \underline{V_R} \cdot \hat{\boldsymbol{n}} dA$$

Example: Moving Cart (Lecture note page 19)

A jet strikes a vane which moves to the right at constant velocity V_C on a frictionless cart. Compute

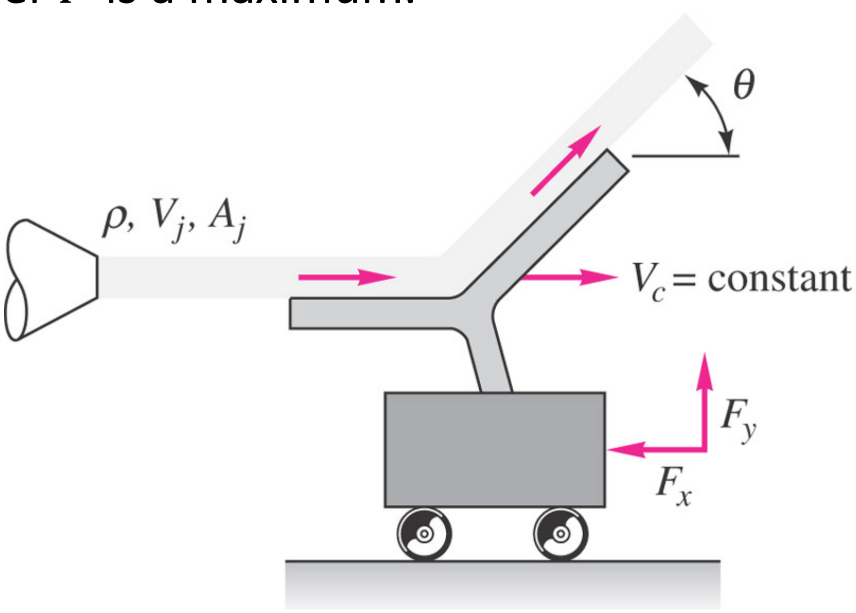
(a) the force F_x required to restrain the cart and

(b) the power P delivered to the cart.

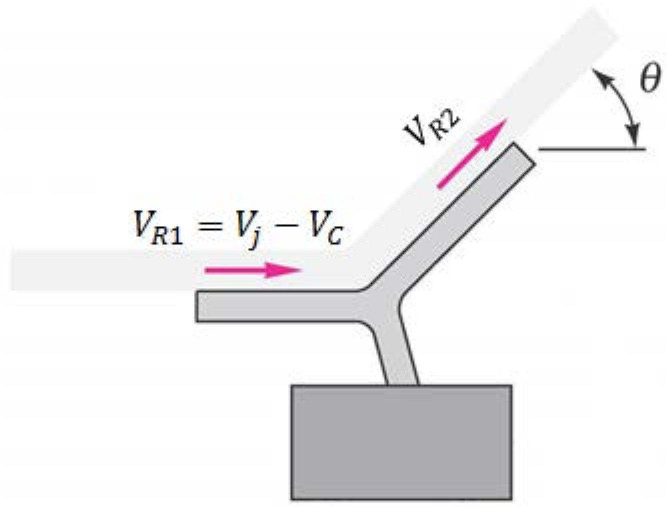
Also find the cart velocity for which

(c) the force F_x is a maximum and

(d) the power P is a maximum.



Bernoulli equation



Moving control volume

Without the elevation terms:

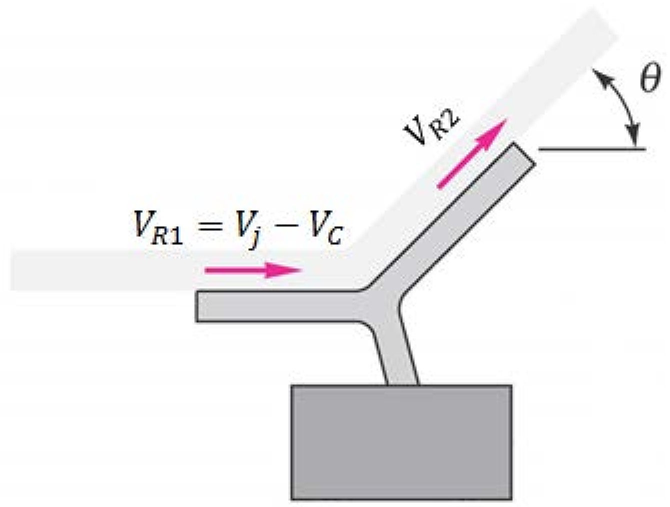
$$\frac{p_1}{\gamma} + \frac{V_{R1}^2}{2g} = \frac{p_2}{\gamma} + \frac{V_{R2}^2}{2g}$$

or

$$\frac{V_{R1}^2}{2g} = \frac{V_{R2}^2}{2g}$$

$$\therefore V_{R1} = V_{R2} = (V_j - V_C)$$

Continuity equation



Moving control volume

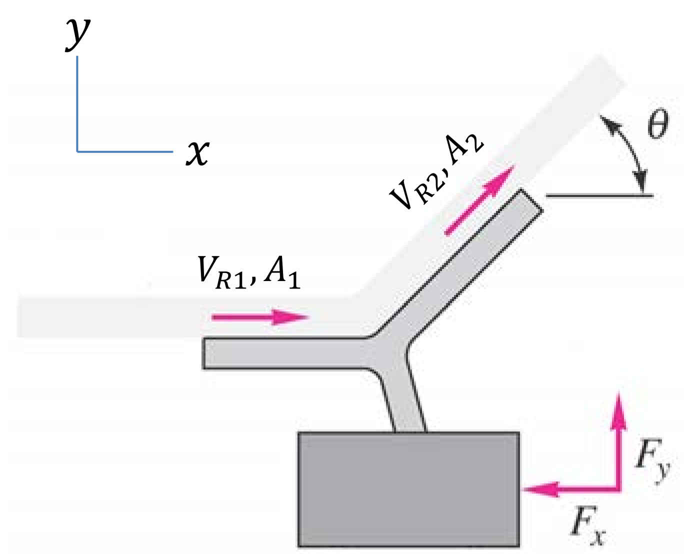
$$0 = \int_{CS} \rho \underline{V_R} \cdot \hat{n} dA$$

or

$$\rho V_{R1} A_1 = \rho V_{R2} A_2$$

$$\therefore A_1 = A_2 = A_j \qquad (\because V_{R1} = V_{R2})$$

Momentum equation



Moving control volume

$$\sum \underline{F} = \int_{CS} \underline{V_R}\rho \underline{V_R} \cdot \hat{n} dA$$

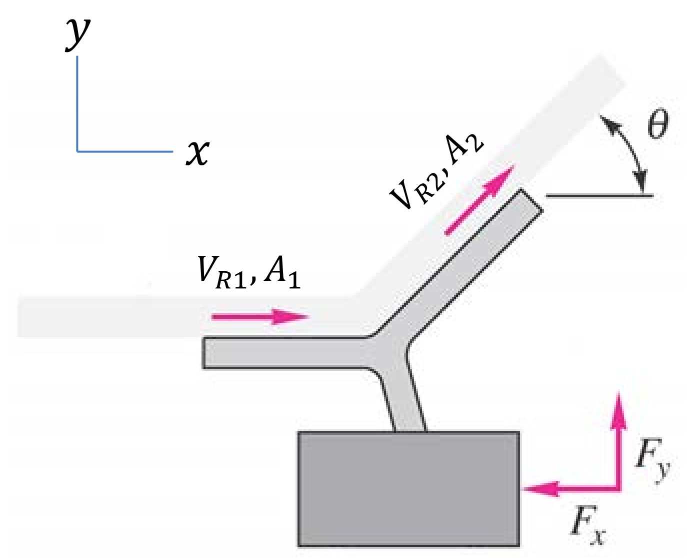
x-component:

$$\sum F_x = (V_{Rx}\rho V_R A)_{out} - (V_{Rx}\rho V_R A)_{in}$$

or

$$-F_x = V_{R2x}(\rho V_{R2} A_2) - V_{R1x}(\rho V_{R_1} A_1)$$

Momentum equation



Moving control volume

$$V_{R1x} = V_{R1} = (V_j - V_C)$$
$$V_{R2x} = V_{R2}\cos\theta = (V_j - V_C)\cos\theta$$

Note:

$$V_{R1} = V_{R2} = (V_j - V_C)$$
$$A_1 = A_2 = A_j$$

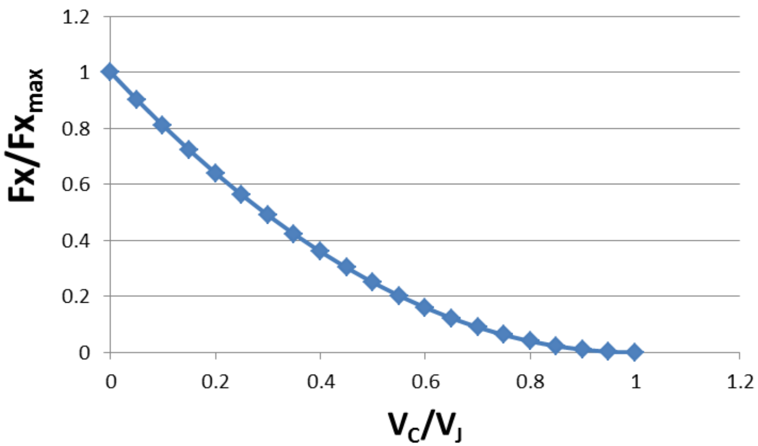
$$-F_x = (V_j - V_C)\cos\theta\,\{\rho(V_j - V_C)A_j\} - (V_j - V_C)\{\rho(V_j - V_C)A_j\}$$

$$\therefore F_x = \rho(V_j - V_C)^2 A_j(1 - \cos\theta)$$

Note: $F_x = \rho V_j^2 A_j(1 - \cos\theta)$ for fixed CV

Maximum F_x

$$F_x(V_C) = \rho\left(V_j - V_C\right)^2 A_j(1 - \cos\theta)$$



Maximum F_x is when V_C = 0:

$$\therefore F_{x_{max}} = \rho V_j^2 A_j(1 - \cos\theta)$$

$$F_x = \underbrace{\rho V_j^2 A_j(1 - \cos\theta)}_{F_{x_{max}}}\left(1 - \frac{V_C}{V_j}\right)^2$$

Power, P

$$P = F_x \cdot V_C$$
$$= \rho V_C \left(V_j - V_C\right)^2 A_j (1 - \cos\theta)$$

Local max/min occurs at V_C that satisfies:

$$\frac{dP}{dV_C} = \rho\left(V_j^2 - 4V_jV_C + 3V_C^2\right)A_j(1 - \cos\theta) = 0$$

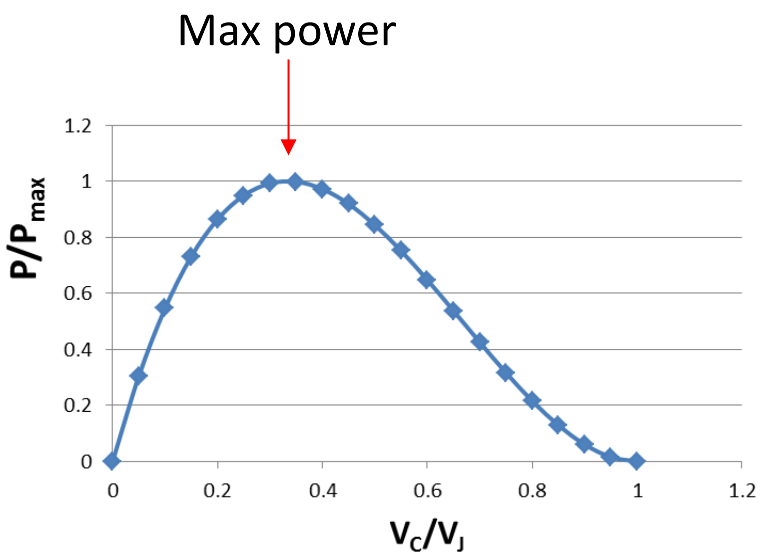
or,

$$3V_C^2 - 4V_CV_j + V_j^2 = 0$$

$$\therefore V_C = \left(\frac{1}{3}V_j, V_j\right)$$

Maximum Power – Contd.

$$P(V_C) = \rho V_C \left(V_j - V_C\right)^2 A_j (1 - \cos\theta)$$



P_{max} is when $V_C = \frac{1}{3} V_j$:

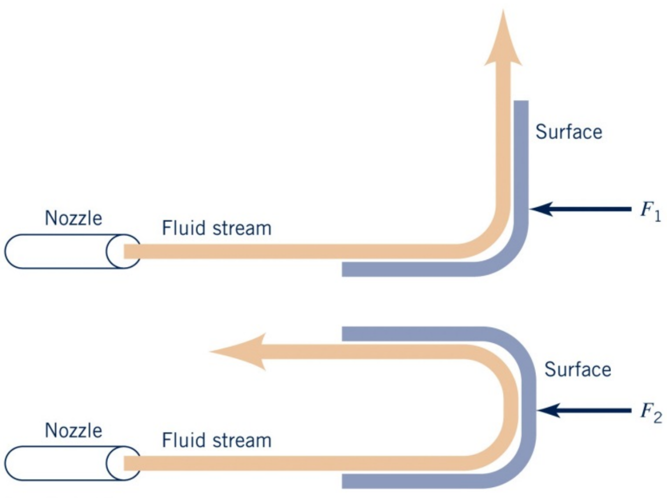
$$\therefore P_{max} = \frac{4}{27} \rho V_j^3 A_j (1 - \cos\theta)$$

$$P = \underbrace{\frac{4}{27} \rho V_j^3 A_j (1 - \cos\theta)}_{P_{max}} \left(\frac{V_C}{V_j}\right) \left(1 - \frac{V_C}{V_j}\right)^2$$

Additional Consideration

At a given V_C:

$$F_x(\theta) = \rho\left(V_j - V_C\right)^2 A_j (1 - \cos\theta)$$



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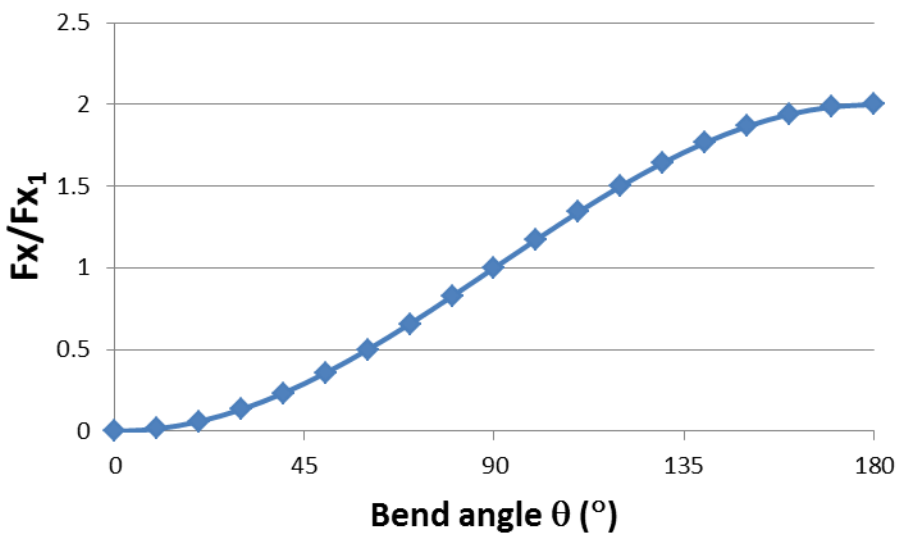


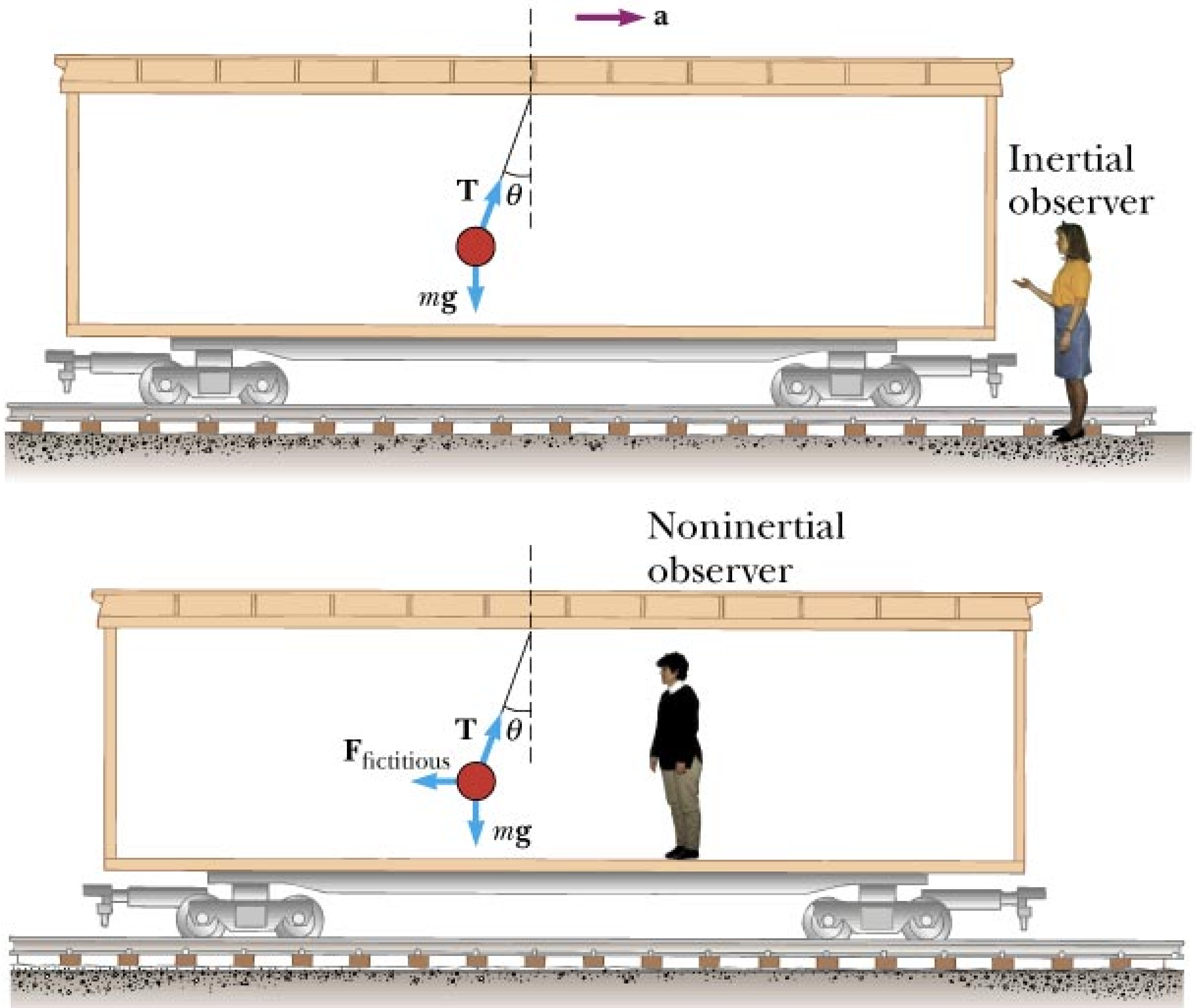
$$F_x = \underbrace{\rho\left(V_j - V_C\right)^2 A_j}_{F_1} (1 - \cos\theta)$$

Inertial vs Non-inertial Frame of Reference

- **Inertial frame of reference**:
  - A frame of reference that is moving uniformly (i.e., at a constant speed without changing its direction or stationary)
  - <u>Laws of mechanics must be same in all inertial frames of reference</u> (Galilean relativity).

- **Non-inertial frame of reference**:
  - A frame of reference that is moving non-uniformly (i.e., accelerating or rotating)
  - The Law of Inertia will appear to be wrong: Need to introduce fictitious forces (e.g., centrifugal force, Coriolis force, Euler force) also called as a phantom force or pseudo force

Example


a
T
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