Applications of Momentum Equation

Initial Setup

1. RTT:

$$\frac{DB_{sys}}{Dt} = \frac{d}{dt}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho \underline{V_R} \cdot d\underline{A}$$

where, $\underline{V_R} = \underline{V} - \underline{V_S}$ and $\underline{dA} = \hat{\boldsymbol{n}} dA$.

If CV is non-deforming and flow is steady ($\partial/\partial t$ = 0),

$$\frac{d}{dt}\int_{CV} \beta\rho d\forall = \int_{CV} \frac{\partial}{\partial t}(\beta\rho) d\forall = 0$$

If CV is fixed (i.e., $\underline{V_S}$=0),

$$\underline{V_R} = \underline{V}$$

Also, if flow is 1D at discrete CS's,

$$\int_{CS} \beta\rho \underline{V} \cdot \underline{dA} = \sum(\beta\dot{m})_{out} - \sum(\beta\dot{m})_{in}$$

where,

$$\dot{m} = \rho\underline{V} \cdot \underline{A} = \rho Q = \rho V A \;\; (V = |\underline{V}| \text{ if } \underline{V} \text{ is normal to } A)$$

2. Linear momentum equation:

Simplified RTT for 1) fixed CV, 2) steady flow, and 3) 1D flows at discrete CS's:

$$\frac{DB_{yss}}{Dt} = \sum(\beta\dot{m})_{out} - \sum(\beta\dot{m})_{in}$$

Let,

$$B = m\underline{V}$$

$$\beta = \frac{dB}{dm} = \underline{V}$$

Then,

$$\frac{DB_{sys}}{Dt} = \sum\underline{F}$$

Thus, a simplified momentum equation becomes

$$\sum\underline{F} = \sum\left(\underline{V}\dot{m}\right)_{out} - \sum\left(\underline{V}\dot{m}\right)_{in}$$

or, in component forms:

$$\sum F_x = \sum(V_x\dot{m})_{out} - \sum(V_x\dot{m})_{in}$$

$$\sum F_y = \sum\left(V_y\dot{m}\right)_{out} - \sum\left(V_y\dot{m}\right)_{in}$$

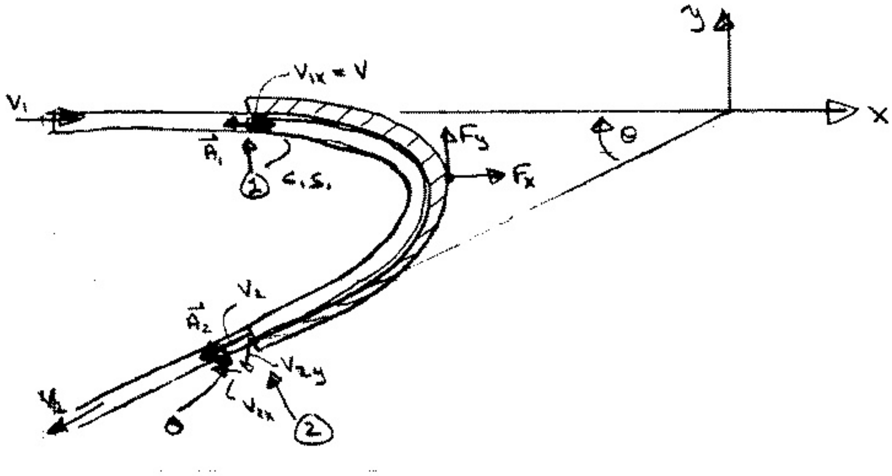
$$\sum F_z = \sum(V_z\dot{m})_{out} - \sum(V_z\dot{m})_{in}$$

where,

$$\dot{m} = \rho Q = \rho VA$$

1. Jet deflected by a plate or vane (Lecture note Ch5, page 12)

Consider a jet of water turned through a horizontal angle



CV and CS are for jet so that F_x and F_y are vane reactions forces on fluid

1) Bernoulli equation

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

Since $p_1=p_2=p_{atm}=0$ (gage) and assume $(V_1^2,V_2^2)\gg 2g(z_1-z_2)$,

$$\therefore V_1=V_2$$

2) Continuity equation

$$\rho_1V_1A_1=\rho_2V_2A_2$$

Since $\rho_1=\rho_2=\rho$ and $V_1=V_2$,

$$\therefore A_1=A_2$$

3) Momentum equation

x-equation:

$$\sum F_x=(V_x\rho VA)_{out}-(V_x\rho VA)_{in}$$

$$F_x=\left(V_{2_x}\right)\left(\rho\underbrace{V_2A_2}_{=Q}\right)-\left(V_{1_x}\right)\left(\rho\underbrace{V_1A_1}_{=Q}\right)$$

$$\therefore F_x=\rho Q\left(V_{2_x}-V_{1_x}\right)$$

y-equation:

$$\sum F_y = (V_y \rho V A)_{out} - (V_y \rho V A)_{in}$$

$$F_y = \left(V_{2_y}\right)\left(\rho \underbrace{V_2 A_2}_{=Q}\right) - \left(V_{1_y}\right)\left(\rho \underbrace{V_1 A_1}_{=Q}\right)$$

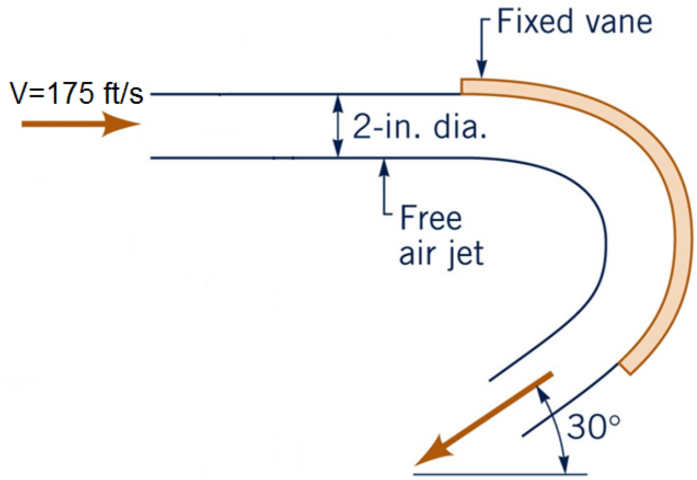
$$\therefore F_y = \rho Q \left(V_{2_y} - V_{1_y}\right)$$

For the given geometry,

$$V_{1_x} = V_1; V_{1y} = 0; V_{2x} = -V_2 \cos\theta\,; V_{2y} = -V_2 \sin\theta$$

Note: F_x and F_y are the forces exerted by the vane on the fluid and $-F_x$ and $-F_y$ are the forces on the vane by the fluid (Newton's 3rd law: Action and reaction)

P5.66-simplified) Air (ρ=2.38×10^{-3} slugs/ft^3) jet of velocity V=175 ft/s strikes a curved vane as shown below. Determine the horizontal component of the anchoring force to hold the vane fixed in place. Neglect the weight of the air and all friction.



x-momentum equation:

$$F_x = (V_x \dot{m})_{out} - (V_x \dot{m})_{in}$$

Since $V_1 = V_2 = V$ from the Bernoulli equation and $A_1 = A_2 = A\ (= \pi D^2/4)$ from the continuity equation,

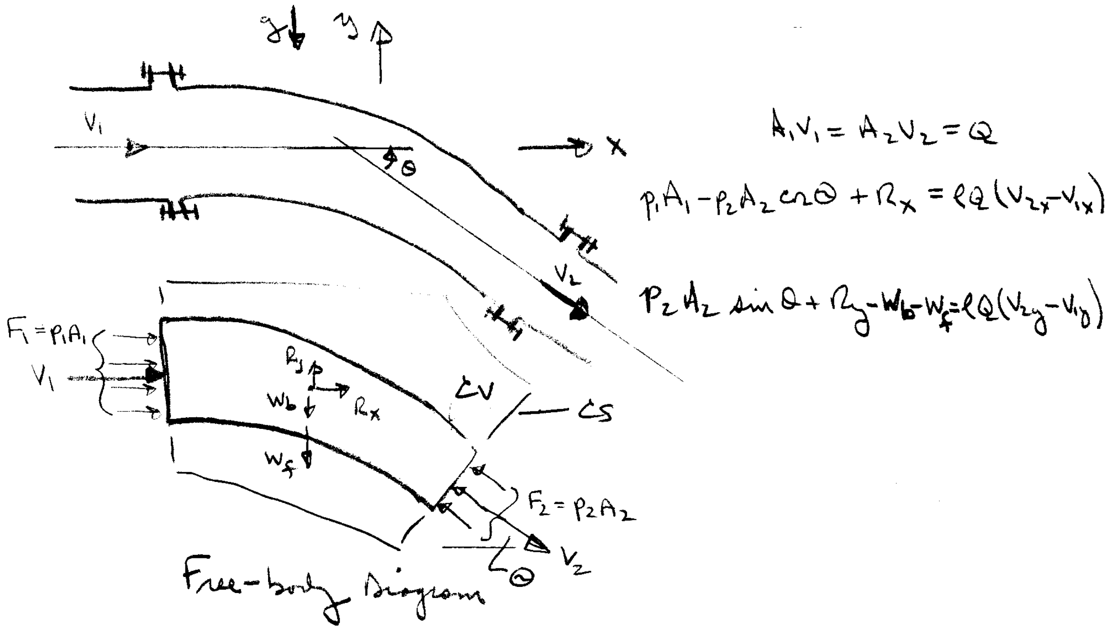
$$F_x = (-V\cos\theta)(\rho V A) - (V)(\rho V A) = -\rho V^2 A(1 + \cos\theta)$$

Thus,

$$F_x = -(2.38 \times 10^{-3})(175)^2 \left(\frac{\pi(2/12)^2}{4}\right)(1 + \cos 30°) = -2.97\ \text{lb}$$

$$\therefore F_x = 2.97\ \text{lb (to the left)}$$

3. <u>Forces on Bends</u> (Lecture note Ch5, page 16)
Consider the flow through a bend in a pipe. The flow is considered steady and uniform across the inlet and outlet sections. Of primary concern is the force required to hold the bend in place, i.e., the reaction forces R_x and R_y which can be determined by application of the momentum equation.



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Continuity equation:

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2$$

Since $\rho_1 = \rho_2 = \rho$ (i.e., incompressible),

$$V_1 A_1 = V_2 A_2 = Q \text{ (= constant)}$$

x-momentum equation:

$$\sum F_x = (V_x \dot{m})_{out} - (V_x \dot{m})_{in}$$

$$p_{1,\text{gage}} A_1 - p_{2,\text{gage}} A_2 \cos\theta + R_x = (V_{x_2})\left(\rho \underbrace{V_2 A_2}_{=Q}\right) - (V_{x_1})\left(\rho \underbrace{V_1 A_1}_{=Q}\right)$$

$$\therefore R_x = \rho Q (V_{x_2} - V_{x_1}) - p_{1,\text{gage}} A_1 + p_{2,\text{gage}} A_2 \cos\theta$$

y-momentum equation:

$$\sum F_y = \left(V_y \dot{m}\right)_{out} - \left(V_y \dot{m}\right)_{in}$$

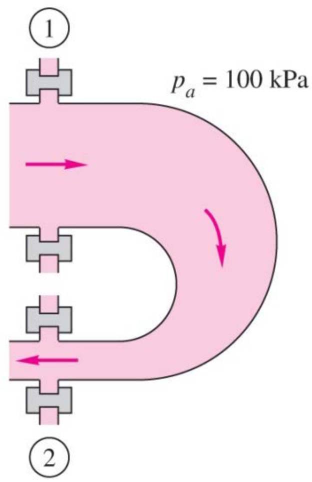
$$-W_f - W_b + p_{2,\text{gage}} A_2 \sin\theta + R_y = \left(V_{y_2}\right)\left(\rho \underbrace{V_2 A_2}_{=Q}\right) - \left(V_{y_1}\right)\left(\rho \underbrace{V_1 A_1}_{=Q}\right)$$

$$\therefore R_x = \rho Q \left(V_{y_2} - V_{y_1}\right) - p_{2,\text{gage}} A_2 \cos\theta + W_f + W_b$$

For the given geometry,

$$V_{1_x} = V_1; V_{1y} = 0; V_{2x} = V_2 \cos\theta\,; V_{2y} = -V_2 \sin\theta$$

Example) Water at 20°C (ρ = 998 kg/m^3) flows steadily through a reducing pipe bend, as shown below. Known conditions are p_1 = 350 kPa, D_1 = 25 cm, V_1 = 2.2 m/s, p_2 = 120 kPa, and D_2 = 8 cm. Neglecting bend and water weight, estimate the total force that must be resisted by the flange bolts.



x-momentum equation:

$$\sum F_x = (V_x \dot{m})_{out} - (V_x \dot{m})_{in}$$

$$p_{1,\mathrm{gage}} A_1 + p_{2,\mathrm{gage}} A_2 + F_B = (-V_2)(\dot{m}_2) - (V_1)(\dot{m}_1)$$

where,

$$\dot{m} = \dot{m}_2 = \dot{m}_1 = \rho A_1 V_1 = (998)\left(\frac{\pi (0.25)^2}{4}\right)(2.2) = 107.8\ \frac{\mathrm{kg}}{\mathrm{m}^3}$$

$$V_2 = \left(\frac{A_1}{A_2}\right) V_1 = \left(\frac{D_1}{D_2}\right)^2 V_1 = \left(\frac{0.25}{0.08}\right)^2 (2.2) = 21.5\ \frac{\mathrm{m}}{\mathrm{s}}$$

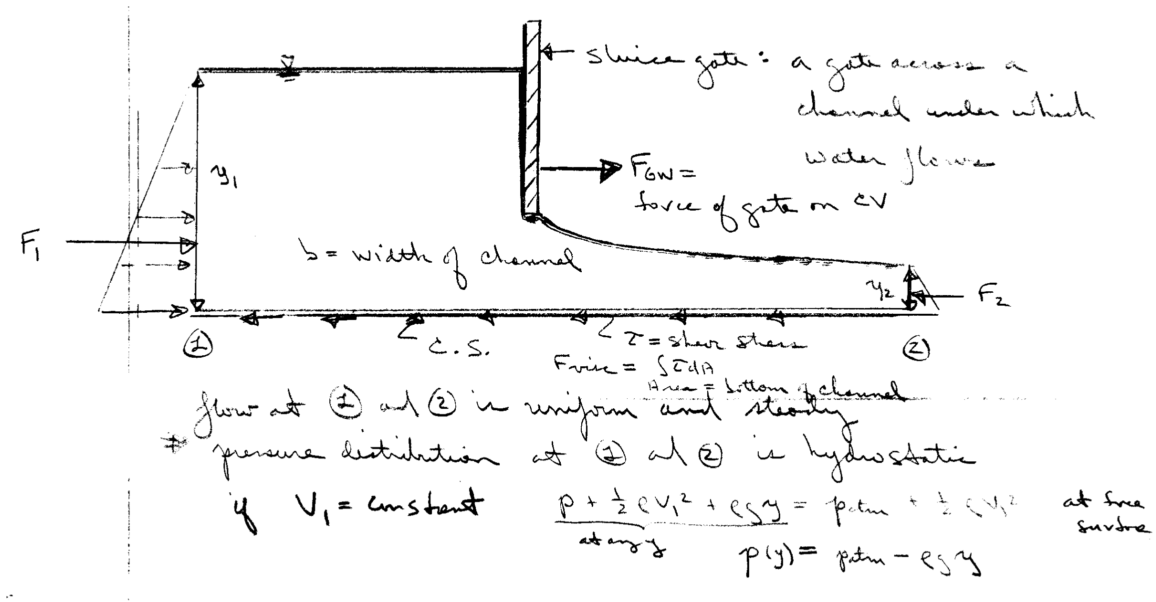
Thus,

$$F_B = -\dot{m}(V_1 + V_2) - p_{1,\mathrm{gage}} A_1 - p_{2,\mathrm{gage}} A_2$$

$$F_B = -(107.8)(2.2 + 21.5)\left(\frac{1}{1000}\right) - (250)\frac{\pi (0.25)^2}{4} - (20)\frac{\pi (0.08)^2}{4} = -14.9\ \mathrm{kN}$$

$$\therefore F_B = 14.9\ \mathrm{kN}\ (\text{to the left})$$

4. <u>Force on a rectangular sluice gate</u> (Lecture note Ch5, page 17) The force on the fluid due to the gate is calculated from the x-momentum equation:



x-momentum equation:

$$\sum F_x = (V_x \dot{m})_{out} - (V_x \dot{m})_{in}$$

$$F_1 - F_2 - F_v + F_{GW} = (V_2)\left(\rho \underbrace{V_2 A_2}_{=Q}\right) - (V_1)\left(\rho \underbrace{V_1 A_1}_{=Q}\right)$$

$$F_{GW} = F_2 - F_1 + \underbrace{F_v}_{\approx 0} + \rho Q (V_2 - V_1)$$

where,

$$F_1 = \overline{p_1} A_1 = \gamma \bar{h} A_1 = \gamma \left(\frac{y_1}{2}\right)(y_1 b) = \frac{\gamma b}{2} y_1^2$$

$$F_2 = \gamma \left(\frac{y_2}{2}\right)(y_2 b) = \frac{\gamma b}{2} y_2^2$$

Thus,

$$F_{GW} = \frac{\gamma b}{2} y_2^2 - \frac{\gamma b}{2} y_1^2 + \rho Q (V_2 - V_1)$$

Note that

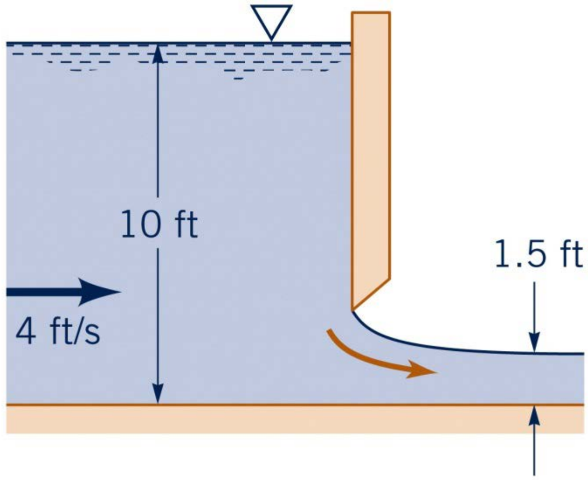
$$V_1 = \frac{Q}{A_1} = \frac{Q}{y_1 b}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{y_2 b}$$

Thus,

$$F_{GW} = \frac{\gamma b}{2}(y_2^2 - y_1^2) + \frac{\rho Q^2}{b}\left(\frac{1}{y_2} - \frac{1}{y_1}\right)$$

P5.62-simplified) Determine the magnitude of the horizontal component of the anchoring force required to hold in place the sluice gate shown below.



Flow rate (per unit width, i.e., let b = 1 ft):

$$Q = V_1 A_1 = (4)(10)(1) = 40\frac{\text{ft}^3}{\text{s}}$$

Gate force (b = 1 ft):

$$F_{GW} = \frac{\gamma b}{2}(y_2^2 - y_1^2) + \frac{\rho Q^2}{b}\left(\frac{1}{y_2} - \frac{1}{y_1}\right)$$

or

$$F_{GW} = \frac{(62.4)(1)}{2}((1.5)^2 - (10)^2) + \frac{(1.94)(40)}{1}\left(\frac{1}{1.5} - \frac{1}{10}\right) = -1290\text{ lb}$$

$$\therefore F_{GW} = 1{,}290\ \text{lb/ft}\quad\text{(to the left)}$$