Review for Exam2

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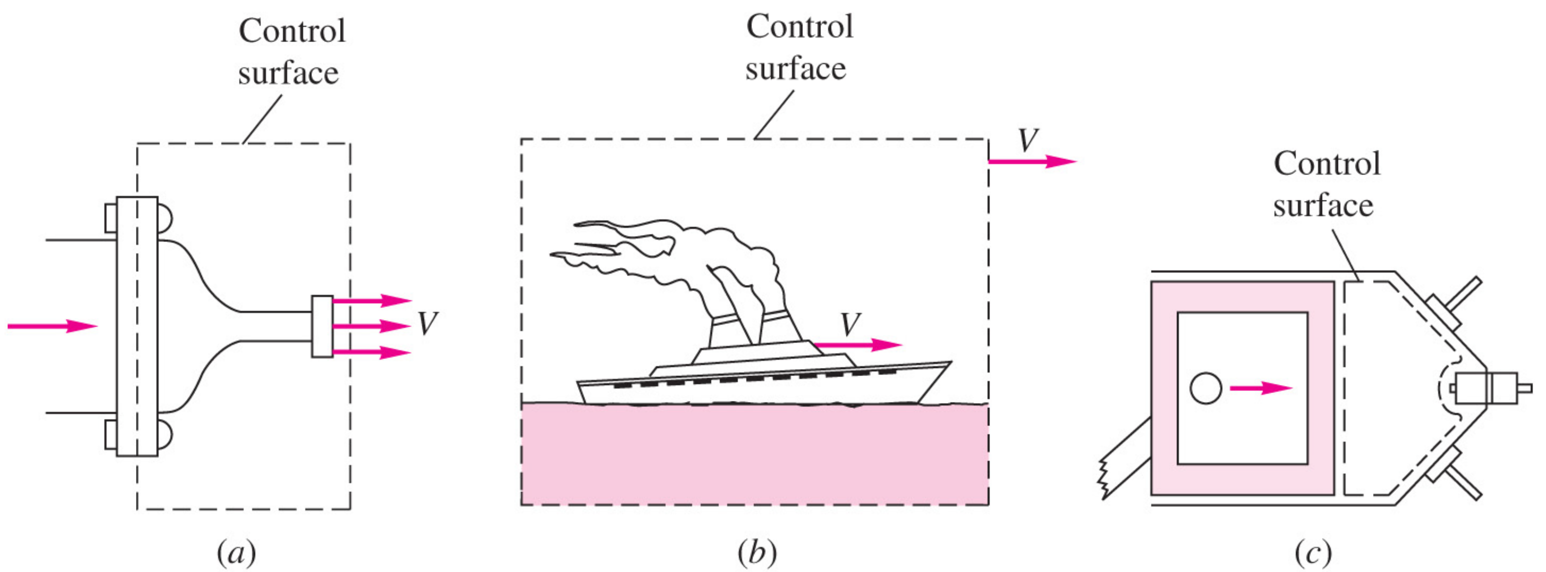
System vs. Control volume

- **System**: A collection of real matter of fixed identity.
- **Control volume (CV)**: A geometric or an imaginary volume in space through which fluid may flow. A CV may move or deform.



Laws of Mechanics for a System

Laws of mechanics are written for a system, i.e., for a fixed amount of matter

- Conservation of mass

$$\frac{Dm}{Dt} = 0$$

- Conservation of momentum

$$\frac{D(m\underline{V})}{Dt} = m\underline{a} = \underline{F}$$

Note:

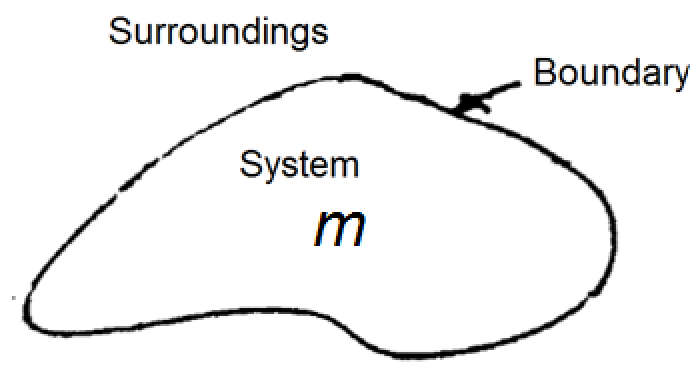
$$\frac{D(m\underline{V})}{Dt} = \underbrace{\frac{Dm}{Dt}}_{=0}\underline{V} + m\underbrace{\frac{D\underline{V}}{Dt}}_{=\underline{a}} = m\underline{a}$$

- Conservation of energy

$$\frac{DE}{Dt} = \dot{Q} - \dot{W}$$

Governing Differential Eq. (GDE):

$$\therefore \frac{D}{Dt}\underbrace{(m, m\underline{V}, E)}_{\substack{\text{system extensive} \\ \text{properties, } B_{sys}}} = \text{RHS}$$



Reynolds Transport Theorem (RTT)

- In fluid mechanics, we are usually interested in a region of space, i.e., CV and not particular systems. Therefore, we need to transform GDE's from a system to a CV, which is accomplished through the use of RTT

$$\underbrace{\frac{DB_{\text{sys}}}{Dt}}_{\substack{\text{time rate of change}\\ \text{of } B \text{ for a system}}} = \underbrace{\frac{D}{Dt}\int_{\text{CV}(\underline{x},t)} \beta\rho d\forall}_{\substack{\text{time rate of change}\\ \text{of } B \text{ in CV}}} + \underbrace{\int_{\text{CS}(\underline{x},t)} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net flux of } B\\ \text{across CS}}}$$

where, $\beta = \frac{dB}{dm} = \left(1, \underline{V}, e\right)$ for $B = (m, m\underline{V}, E)$

- Fixed CV,

$$\frac{DB_{\text{sys}}}{Dt} = \frac{\partial}{\partial t}\int_{\text{CV}} \beta\rho d\forall + \int_{\text{CS}} \beta\rho \underline{V} \cdot d\underline{A}$$

Note:

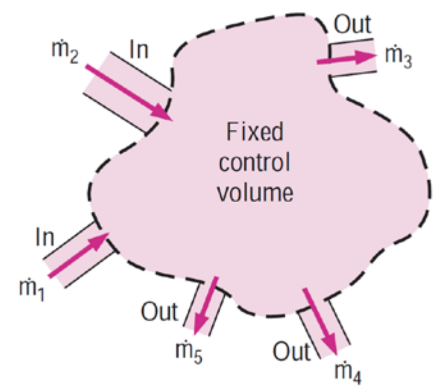
$$B_{\text{CV}} = \int_{\text{CV}} \beta dm = \int_{\text{CV}} \beta\rho d\forall$$

$$\dot{B}_{\text{CS}} = \int_{\text{CS}} \beta d\dot{m} = \int_{\text{CS}} \beta\rho \underline{V} \cdot d\underline{A}$$

Continuity Equation

- RTT with $B = m$ and $\beta = 1$,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho d\forall + \int_{\mathrm{CS}} \rho \underline{V} \cdot d\underline{A} = 0$$



- Steady flow,

$$\int_{\mathrm{CS}} \rho \underline{V} \cdot d\underline{A} = 0$$

- Simplified form,

$$\sum \dot{m}_{\mathrm{out}} - \sum \dot{m}_{\mathrm{in}} = 0$$

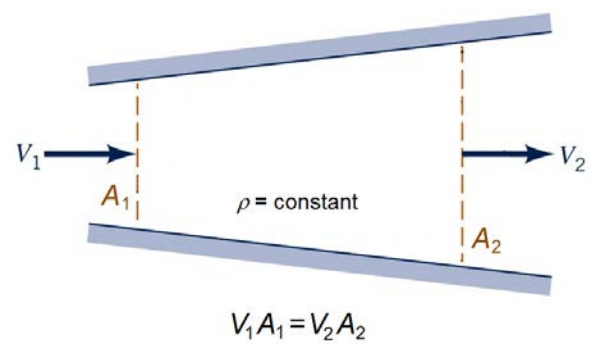
Note: $\dot{m} = \rho Q = \rho V A$

- Conduit flow with one inlet (1) and one outlet (2):

$$\rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0$$

If ρ = constant,

$$V_1 A_1 = V_2 A_2$$



Momentum Equation

- RTT with $B = m\underline{V}$ and $\beta = \underline{V}$,

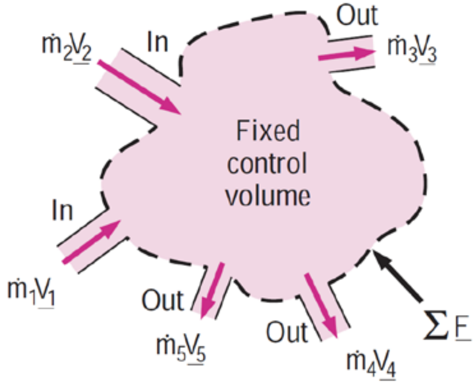
$$\frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho\underline{V}\cdot d\underline{A} = \sum \underline{F}$$

- Simplified form:

$$\sum(\dot{m}\underline{V})_{out} - \sum(\dot{m}\underline{V})_{in} = \sum \underline{F}$$

or in component forms,

$$\begin{aligned}
\sum(\dot{m}u)_{out} - \sum(\dot{m}u)_{in} &= \sum F_x \\
\sum(\dot{m}v)_{out} - \sum(\dot{m}v)_{in} &= \sum F_y \\
\sum(\dot{m}w)_{out} - \sum(\dot{m}w)_{in} &= \sum F_z
\end{aligned}$$



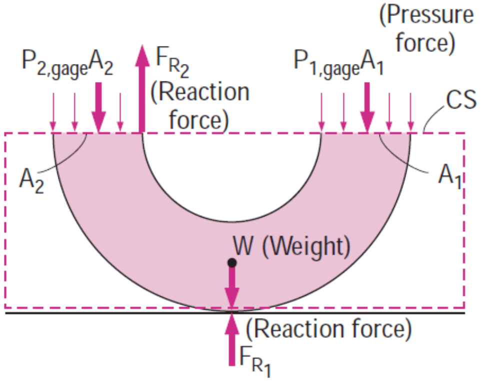
Note: If $\underline{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{\boldsymbol{k}}$ is normal to CS, $\dot{m} = \rho V A$, where $V = |\underline{V}|$.

Momentum Equation – Contd.

- External forces:

$$\sum \underline{F} = \sum \underline{F}_{\text{body}} + \sum \underline{F}_{\text{surface}} + \sum \underline{F}_{\text{other}}$$

  - $\sum \underline{F}_{\text{body}} = \sum \underline{F}_{\text{gravity}}$
    - $\sum \underline{F}_{\text{gravity}}$: gravity force (i.e., weight)
  - $\sum \underline{F}_{\text{Surface}} = \sum \underline{F}_{\text{pressure}} + \sum \underline{F}_{\text{friction}} + \sum \underline{F}_{\text{other}}$
    - $\sum \underline{F}_{\text{pressure}}$: pressure forces normal to CS
    - $\sum \underline{F}_{\text{friction}}$: viscous friction forces tangent to CS
  - $\sum \underline{F}_{\text{other}}$: anchoring forces or reaction forces

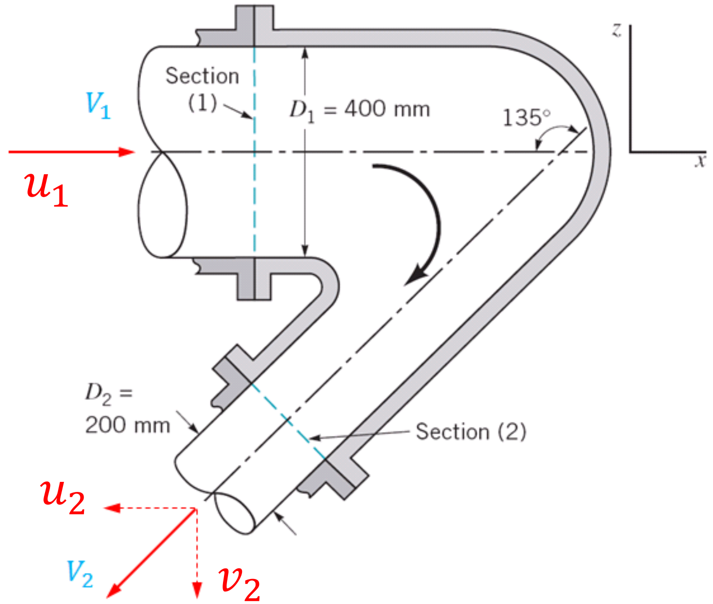


An 180° elbow supported by the ground

In most flow systems, the force $\vec{F}$ consists of weights, pressure forces, and reaction forces. Gage pressures are used here since atmospheric pressure cancels out on all sides of the control surface.

Note: Surface forces arise as the CV is isolated from its surroundings, similarly to drawing a free-body diagram. A well-chosen CV exposes only the forces that are to be determined and a minimum number of other forces

Example (Bend)



Inlet (1):

$$\dot{m}_1 = \rho V_1 A_1$$
$$u_1 = V_1$$
$$v_1 = 0$$

Outlet (2):

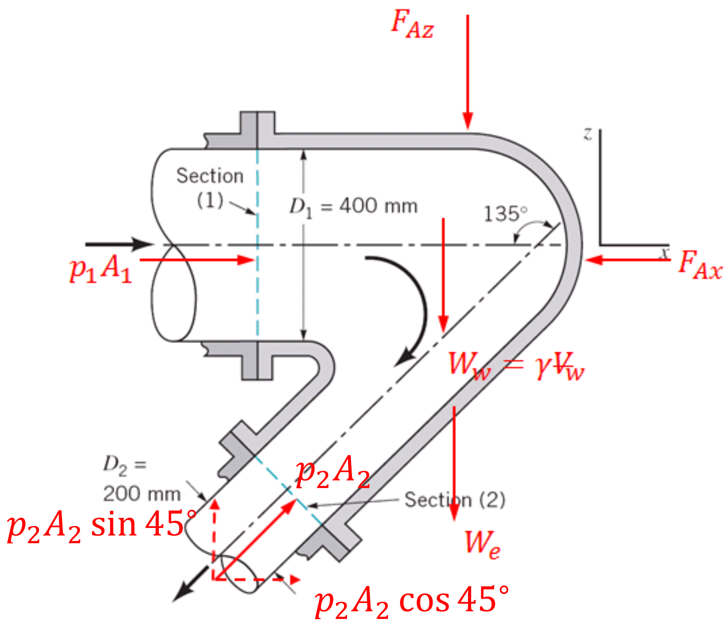
$$\dot{m}_2 = \rho V_2 A_2$$
$$u_2 = -V_2 \cos 45°$$
$$v_2 = -V_2 \sin 45°$$

$$(\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}} = (\rho V_2 A_2)(-V_2 \cos 45°) - (\rho V_1 A_1)(V_1)$$
$$(\dot{m}v)_{\text{out}} - (\dot{m}v)_{\text{in}} = (\rho V_2 A_2)(-V_2 \sin 45°) - (\rho V_1 A_1)(0)$$

Since $\rho V_1 A_1 = \rho V_2 A_2$,

$$(\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}} = -(\rho V_2 A_2)(V_2 \cos 45° + V_1)$$
$$(\dot{m}v)_{\text{out}} - (\dot{m}v)_{\text{in}} = -\rho V_2^2 A_2 \sin 45°$$

Example – Contd.



$\sum F_x$:

1) Body force = 0
2) Pressure force = $p_1A_1 + p_2A_2\cos 45°$
3) Anchoring force = $-F_{Ax}$

$\sum F_y$:

1) Body force = $-W_w - W_e$
2) Pressure force = $p_2A_2\sin 45°$
3) Anchoring force = $-F_{Az}$

Thus,

$$-(\rho V_2A_2)(V_2\cos 45° + V_1) = p_1A_1 + p_2A_2\cos 45° - F_{Ax}$$
$$-\rho V_2^2A_2\sin 45° = -\gamma\forall_w - W_e + p_2A_2\sin 45° - F_{Az}$$

$$\therefore F_{Ax} = (\rho V_2A_2)(V_2\cos 45° + V_1) + p_1A_1 + p_2A_2\cos 45°$$
$$F_{Az} = \rho V_2^2A_2\sin 45° - \gamma\forall_w - W_e + p_2A_2\sin 45°$$

Energy Equation

- RTT with $B = E$ and $\beta = e$,

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} e\rho d\forall + \int_{\mathrm{CS}} e\rho \underline{V} \cdot d\underline{A} = \dot{Q} - \dot{W}$$

- Simplified form:

$$\frac{p_{\mathrm{in}}}{\gamma} + \alpha_{\mathrm{in}}\frac{V_{\mathrm{in}}^2}{2\mathrm{g}} + z_{\mathrm{in}} + h_p = \frac{p_{\mathrm{out}}}{\gamma} + \alpha_{\mathrm{out}}\frac{V_{\mathrm{out}}^2}{2\mathrm{g}} + z_{\mathrm{out}} + h_t + h_L$$

- V in energy equation refers to average velocity $\bar{V}$

- α : kinetic energy correction factor $= \begin{cases} 1 \text{ for uniform flow across CS} \\ 2 \text{ for laminar pipe flow} \\ \approx 1 \text{ for turbulent pipe flow} \end{cases}$

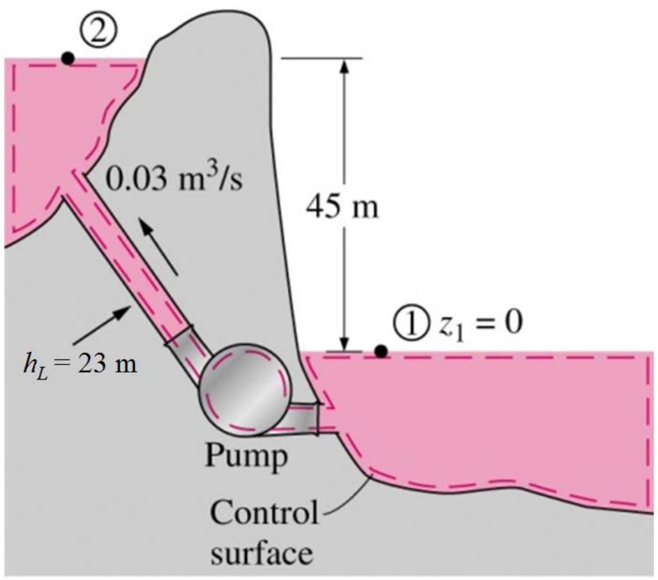
Energy Equation - Contd.

Uniform flow across CS's:

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2\text{g}}+z_1+h_p=\frac{p_2}{\gamma}+\frac{V_2^2}{2\text{g}}+z_1+h_t+h_L$$

- Pump head $h_p=\frac{\dot{W}_p}{\dot{m}\text{g}}=\frac{\dot{W}_p}{\rho Q\text{g}}=\frac{\dot{W}_p}{\gamma Q} \Rightarrow \dot{W}_p=\dot{m}\text{g}h_p=\rho\text{g}Qh_p=\gamma Qh_p$
- Turbine head $h_t=\frac{\dot{W}_t}{\dot{m}\text{g}}=\frac{\dot{W}_t}{\rho Q\text{g}}=\frac{\dot{W}_t}{\gamma Q} \Rightarrow \dot{W}_t=\dot{m}\text{g}h_t=\rho\text{g}Qh_t=\gamma Qh_t$
- Head loss $h_L=\text{loss}/\text{g}=(\hat{u}_2-\hat{u}_1)/\text{g}-\dot{Q}/\dot{m}\text{g}>0$

Example (Pump)



Energy equation:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

With $p_1 = p_2 = 0$, $V_1 = V_2 \approx 0$, $h_t = 0$, and $h_L = 23$ m

$$h_p = (z_2 - z_1) + h_L = 45 + 23 = 68 \text{ m}$$

Pump power,

$$\dot{W}_p = \gamma Q h_p = \frac{(68)(9790)(0.03)}{746} = 80 \text{ hp}$$

(Note: 1 hp = 746 N·m/s = 550 ft·lbf/s)

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.

Differential Analysis

A microscopic description of fluid motions for a fluid particle by using differential equations*,

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{V}) = 0$$

- Momentum equation

$$\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V}\right) = \rho \underline{\mathrm{g}} - \nabla p + \nabla \cdot \tau_{ij}$$

*CV analysis is a macroscopic description of fluid motions by using integral equations (RTT).

Navier-Stokes Equations

For incompressible, Newtonian fluids,

- Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Momentum:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho \mathrm{g}_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho \mathrm{g}_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho \mathrm{g}_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Exact Solutions of NS Eqns.

The flow of interest is assumed additionally (than incompressible & Newtonian), for example,

1) Steady (i.e., $\boldsymbol{\partial/\partial t = 0}$ for any variable)
2) Parallel such that the y-component of velocity is zero (i.e., $\boldsymbol{v = 0}$)
3) Purely two dimensional (i.e., $\boldsymbol{w = 0}$ and $\boldsymbol{\partial/\partial z = 0}$ for any velocity component)
4) Fully developed (i.e., $\boldsymbol{\partial/\partial x = 0}$ for any velocity component)

e.g.)

$$\rho\left[\overbrace{\frac{\partial u}{\partial t}}^{1)} + u\overbrace{\frac{\partial u}{\partial x}}^{4)} + \overbrace{v}^{2)}\frac{\partial u}{\partial y} + \overbrace{w\frac{\partial u}{\partial z}}^{3)}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\overbrace{\frac{\partial^2 u}{\partial x^2}}^{4)} + \frac{\partial^2 u}{\partial y^2} + \overbrace{\frac{\partial^2 u}{\partial z^2}}^{3)}\right]$$

or

$$\mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial x} - \rho g_x$$

Boundary Conditions

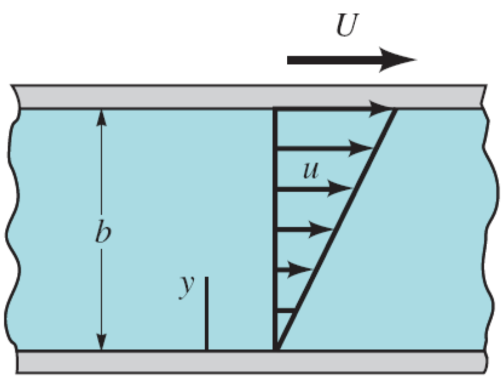
Common BC's:

- **No-slip condition** ($\underline{V}_{\text{fluid}} = \underline{V}_{\text{wall}}$; for a stationary wall $\underline{V}_{\text{fluid}} = 0$)
- Interface boundary condition ($\underline{V}_A = \underline{V}_B$ and $\tau_{s,A} = \tau_{s,B}$)
- Free-surface boundary condition ($p_{\text{liquid}} = p_{\text{gas}}$ and $\tau_{s,\text{liquid}} = 0$)

Other BC's:

- Inlet/outlet boundary condition
- Symmetry boundary condition
- Initial condition (for unsteady flow problem)

Example: No pressure gradient



$$\mu \frac{d^2u}{dy^2} = 0$$

Integrate twice,

$$u(y) = C_1 y + C_2$$

B.C.,

$$u(0) = (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$u(b) = (C_1)(b) + C_2 = U \quad \Rightarrow \quad C_1 = \frac{U}{b}$$

$$\therefore u(y) = \frac{U}{b} y$$

Analysis:

$$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = (\mu)\left(\frac{U}{b}\right) = \frac{\mu U}{b}$$

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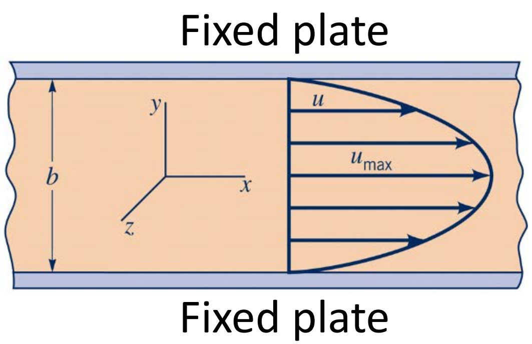
Example: with Pressure Gradient



$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx}$$

Integrate twice,

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

B.C.,

$$u(0) = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)(0)^2 + (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$u(b) = \left(\frac{1}{2\mu}\frac{dp}{dx}\right)(b)^2 + (C_1)(b) + C_2 = 0 \Rightarrow \quad C_1 = -\frac{1}{2\mu}\frac{dp}{dx}b$$

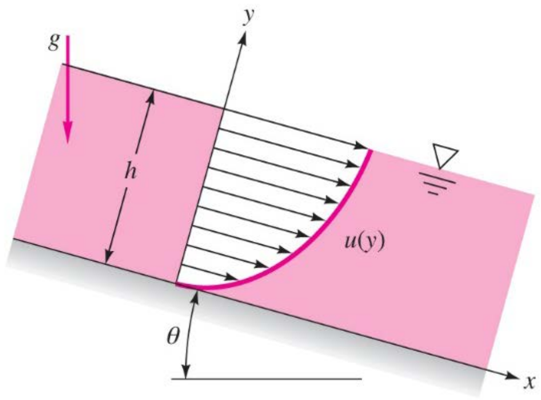
$$\therefore u(y) = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)(y^2 - by)$$

Analysis:

$$q = \int_{-h}^{h} u dy = -\frac{b^3}{12\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\tau_w = \mu \frac{du}{dy}\bigg)_{y=0} = -\frac{b}{2}\left(\frac{\partial p}{\partial x}\right)$$

Example: Inclined wall



$$\mu \frac{d^2u}{dy^2} = -\rho g_x$$

Integrate twice,

$$u(y) = -\frac{\rho g_x}{2\mu} y^2 + C_1 y + C_2$$

B.C.,

$$u(0) = \left(-\frac{\rho g_x}{\mu}\right)(0)^2 + (C_1)(0) + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

$$\left.\frac{du}{dy}\right)_{y=h} = \left(-\frac{\rho g_x}{\mu}\right)(h) + C_1 = 0 \quad \Rightarrow \quad C_1 = \frac{\rho g_x}{\mu} h$$

Note:

$\underline{g} = g_x \hat{\imath} + g_y \hat{\jmath}$

where,

$g_x = g \sin\theta$

$g_y = -g \cos\theta$

$$\therefore u(y) = \frac{\rho g_x}{\mu}\left(hy - \frac{y^2}{2}\right)$$

Analysis:

$$q = \int_0^h u dy = \frac{\rho g_x}{\mu} \frac{h^3}{3}$$

$$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = (\mu)\left(\frac{\rho g_x}{\mu} h\right) = \rho g_x h$$

Buckingham Pi Theorem

- For any physically meaningful equation involving $\boldsymbol{n}$ **variables**, such as

$$u_1 = f(u_2, u_3, \cdots, u_n)$$

with minimum number of $\boldsymbol{m}$ **reference dimensions**, the equation can be rearranged into product of $\boldsymbol{r}$ **dimensionless pi terms**.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_r)$$

where,

$$\boldsymbol{r = n - m}$$

Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

For,

$$\Pi_1 = \phi(\Pi_2, \ldots, \Pi_n)$$

Similarity requirements:

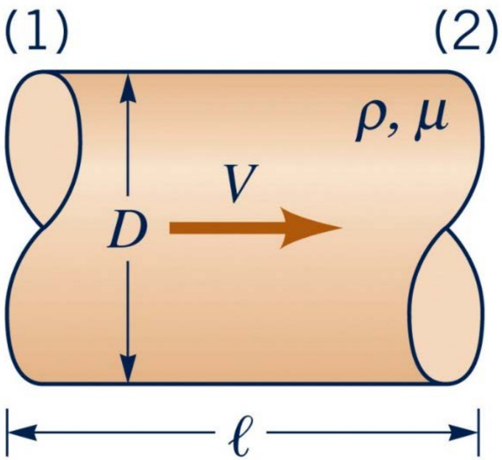
$$\Pi_{2,\text{model}} = \Pi_{2,\text{prototype}}$$
$$\vdots$$
$$\Pi_{n,\text{model}} = \Pi_{n,\text{prototype}}$$

Prediction equation:

$$\Pi_{1,\text{model}} = \Pi_{1,\text{prototype}}$$

Example (Repeating Variable Method)

Example: The pressure drop per unit length Δp_ℓ in a pipe flow is a function of the pipe diameter D and the fluid density ρ, viscosity μ, and velocity V.



$\Delta p_\ell = (p_1 - p_2)/\ell$

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Δp_ℓ	D	ρ	μ	V
$\{ML^{-2}T^{-2}\}$	$\{L\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$	$\{LT^{-1}\}$

$$r = n - m = 5 - 3 = 2$$

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Example – Contd.

Select m = 3 repeating variables, (D, V, ρ) for (L, T, M), then

$$\Pi_1 = D^a V^b \rho^c \Delta p_\ell \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-2}T^{-2}) \doteq M^0 L^0 T^0$$

$$\begin{array}{rcl} a + b - 3c - 2 & = & 0 \\ -b - 2 & = & 0 \\ c + 1 & = & 0 \end{array} \quad \Rightarrow \; a = -1, b = -2, c = -1$$

$$\Rightarrow \Pi_1 = D^{-1} V^{-2} \rho^{-1} \Delta p_\ell = \frac{\Delta p_\ell D}{\rho V^2}$$

$$\Pi_2 = D^a V^b \rho^c \mu \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1}) \doteq M^0 L^0 T^0$$

$$\begin{array}{rcl} a + b - 3c - 1 & = & 0 \\ -b - 1 & = & 0 \\ c + 1 & = & 0 \end{array} \quad \Rightarrow \; a = -1, b = -1, c = -1$$

$$\Rightarrow \Pi_2 = D^{-1} V^{-1} \rho^{-1} \mu = \frac{\mu}{DV\rho}$$

$$\therefore \frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

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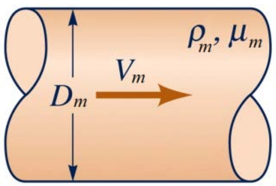
Example (Model Testing)

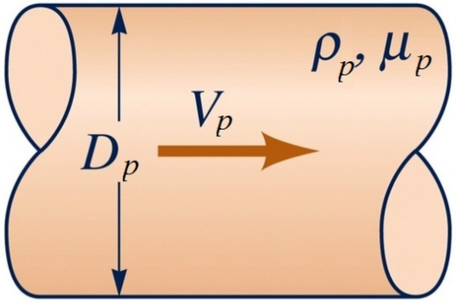
Model



Δp_{ℓ_m}

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Prototype



Δp_{ℓ_p}

If,

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$$ (similarity requirement)

Then,

$$\frac{\Delta p_{\ell_m} D_m}{\rho_m V_m^2} = \frac{\Delta p_{\ell_p} D_p}{\rho_p V_p^2}$$ (Prediction equation)

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Example – Contd.

Model (in water)

- D_m = 0.1 m
- ρ_m = 998 kg/m^3
- μ_m = 1.12 × 10^{-3} N·s/m^2
- V_m = ?
- Δp_{ℓ_m} = 27.6 Pa/m

Prototype (in air)

- D_p = 1 m
- ρ_p = 1.23 kg/m^3
- μ_p = 1.79 × 10^{-5} N·s/m^2
- V_p = 10 m/s
- Δp_{ℓ_m} = ?

Similarity requirement:

$$V_m = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{D_p}{D_m}\right)V_p = \left(\frac{1.23}{998}\right)\left(\frac{1.12\times 10^{-3}}{1.79\times 10^{-5}}\right)\left(\frac{1}{0.1}\right)(10) = \mathbf{7.71\ m/s}$$

Prediction equation:

$$\Delta p_{\ell_p} = \left(\frac{D_m}{D_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2 \Delta p_{\ell_m} = \left(\frac{0.1}{1}\right)\left(\frac{1.23}{998}\right)\left(\frac{10}{7.71}\right)^2(27.6) = \mathbf{5.72\times 10^{-3}\ Pa/m}$$

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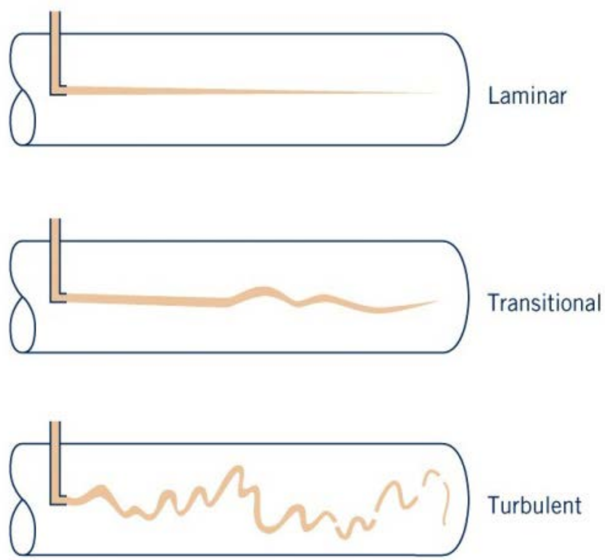
Pipe Flow: Laminar vs. Turbulent

- Reynolds number regimes

$$\mathrm{Re} = \frac{\rho V D}{\mu}$$



$$\mathrm{Re} < \mathrm{Re}_{\mathrm{crit}} \sim 2{,}000$$

$$\mathrm{Re}_{\mathrm{crit}} < \mathrm{Re} < \mathrm{Re}_{\mathrm{trans}}$$

$$\mathrm{Re} > Re_{\mathrm{trans}} \sim 4{,}000$$

Flow in Pipes

- Basic piping problems:
  - Given the desired flow rate, what pressure drop (e.g., pump power) is needed to drive the flow (i.e., to overcome the head loss through piping)?
  - Given the pressure drop (e.g., pump power) available, what flow rate will ensue?
  - Given the pressure drop and the flow rate desired, what pipe diameter is needed?

Head Loss

$$h_L = h_{L\,\text{major}} + h_{L\,\text{minor}}$$

- $h_{L\,\text{major}}$ (or h_f): Major loss, the loss due to viscous effects
- $h_{L\,\text{minor}}$: Minor loss, the loss in the various pipe components

Darcy-Weisbach equation

$$h_f = f\frac{L}{D}\frac{V^2}{2\text{g}}$$

- $f = \frac{8\tau_w}{\rho V^2}$: Friction factor
- L: Pipe length
- D: Pipe diameter
- V: Average flow velocity across the pipe cross-section

Laminar Pipe Flow

- Exact solution exists by solving the NS equation

$$u(r) = V_{\max}\left[1 - \left(\frac{r}{R}\right)^2\right], \qquad V_{\max} = 2V$$

- Wall shear stress

$$\tau_w = -\mu \left.\frac{du}{dr}\right)_{r=R} = \frac{8\mu V}{D}$$

- Friction factor

$$f = \frac{8\tau_w}{\rho V^2} = \frac{64\mu}{\rho D V} = \frac{64}{\mathrm{Re}}$$

- Head loss

$$h_f = f\frac{L}{D}\frac{V^2}{2\mathrm{g}} = \frac{32\mu L V}{\gamma D^2} \quad \left(= \frac{128\mu L Q}{\pi\gamma D^4}\right)$$

Notes:

$$Q = VA$$

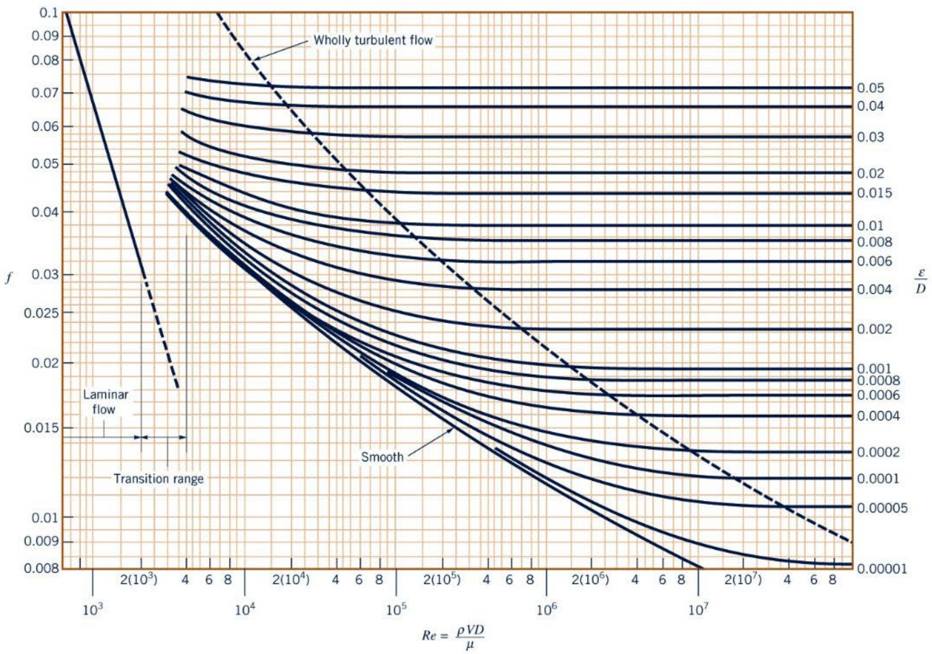
$$A = \frac{\pi D^2}{4}$$

Turbulent Pipe Flow

- From a dimensional analysis

$$f = \phi(\mathrm{Re}, \varepsilon/D)$$

- Moody chart: Empirical functional dependency of f on Re and ε/D



Turbulent Pipe Flow – Cond.

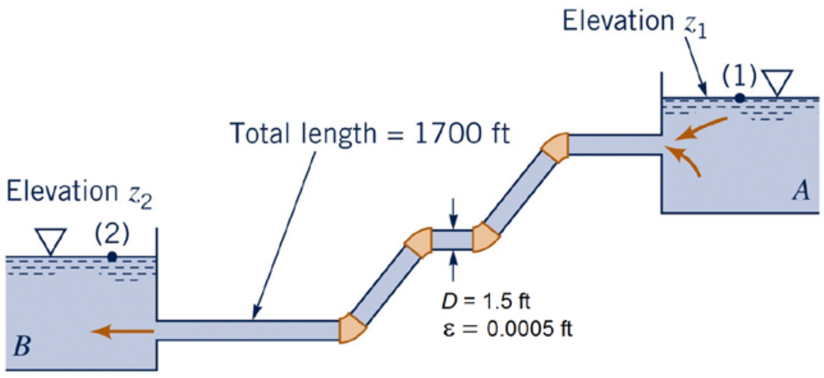
- Colebrook equation (difficult in its use as implicit)

$$\frac{1}{\sqrt{f}} = -2\log\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right)$$

- Haaland equation (easier to use as explicit but approximation)

$$\frac{1}{\sqrt{f}} = -1.8\log\left[\left(\frac{\varepsilon/D}{3.7}\right)^{1.1} + \frac{6.9}{Re}\right]$$

Example (pipe flow)



If D = 1.5 ft and Q = 25 ft^3/s, $\Delta z = z_1 - z_2$?
Neglect minor losses.

$$V = \frac{Q}{A} = \frac{25}{(\pi)(1.5)^2/4} = 14.1\ \text{ft/s}$$

$$\text{Re} = \frac{VD}{\nu} = \frac{(14.1)(1.5)}{1.21 \times 10^{-5}} = 1.75 \times 10^6\ \text{(turbulent)}$$

$$\varepsilon/D = 0.0005/1.63 = 0.00033$$

Energy equation:

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_f$$

Since $p_1 = p_2 = 0$ and $V_1 = V_2$,

$$h_f = z_1 - z_2 = \Delta z$$

Friction factor,

$$\frac{1}{\sqrt{f}} = -1.8 \log\left[\left(\frac{0.00033}{3.7}\right)^{1.1} + \frac{6.9}{1.75 \times 10^6}\right] \quad \Rightarrow \quad f = 0.0159$$

Head loss

$$h_f = f \frac{L}{D} \frac{V^2}{2g} = (0.0159) \frac{(1700)}{(1.5)} \frac{(14.1)^2}{(2)(32.2)} = 56\ \text{ft}$$

$$\therefore \Delta z = \mathbf{56\ ft}$$

This slide contains an example problem and its contents (except for general formula) should NOT be included in your cheat-sheet.