Review for Exam2

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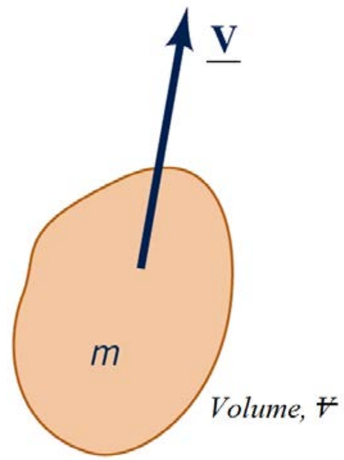
Mass and Momentum

For a homogeneous body of volume $\forall$,

$$\text{Mass,} \qquad m = \rho \forall$$

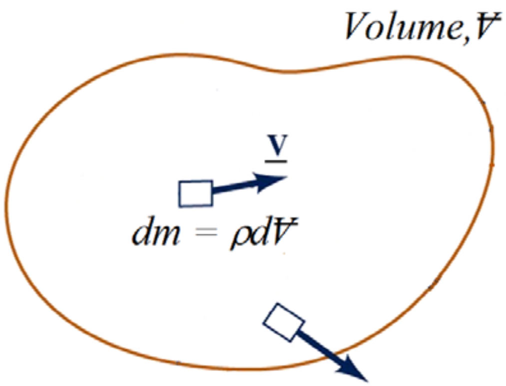
$$\text{Momentum,} \qquad \underline{\text{P}} = m\underline{V}$$



For a fluid of volume $\forall$,

$$m = \int_{\forall} dm = \int_{\forall} \rho dV$$

$$\underline{\text{P}} = \int_{\forall} \underline{V} dm = \int_{\forall} \underline{V} \rho dV$$



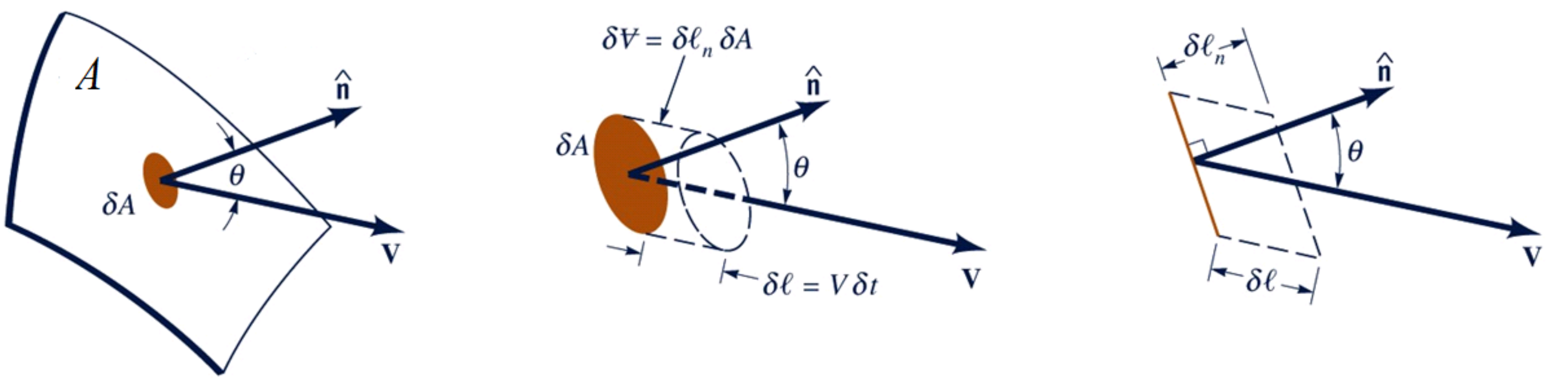
Mass Flux

The amount of fluid mass flowing through a small portion of a surface δA during a short time δt,

$$\delta m = \rho \delta \forall = \rho \delta \ell_n \delta A = \rho (V \delta t \cos\theta) \delta A = \rho \left(\underline{V} \cdot \hat{n} \right) \delta t \delta A$$

Then, the time rate of δm becomes

$$\delta \dot{m} = \frac{\delta m}{\delta t} = \frac{\rho \left(\underline{V} \cdot \hat{n} \right) \delta t \delta A}{\delta t} = \rho \left(\underline{V} \cdot \hat{n} \right) \delta A$$



(a) (b) (c)

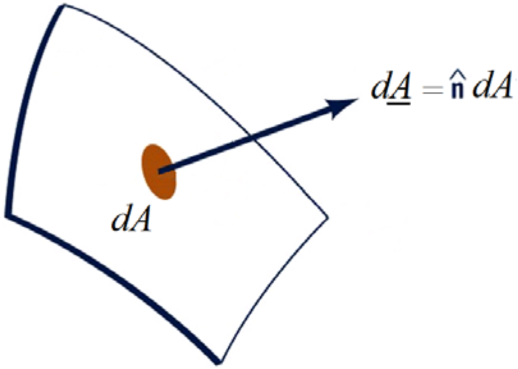
Mass Flux – Contd.

- The rate of mass of fluid flowing through an area A:

$$\dot{m} = \int_A d\dot{m}$$

Where,

$$d\dot{m} = \lim_{\delta t, \delta A \to 0} \frac{\rho(\underline{V} \cdot \hat{n})\delta t \delta A}{\delta t} = \rho(\underline{V} \cdot \hat{n}) dA = \rho \underline{V} \cdot d\underline{A}$$



Therefore,

$$\dot{m} = \int_A \rho \underline{V} \cdot d\underline{A}$$

If ρ = constant,

$$\dot{m} = \rho \int_A \underline{V} \cdot d\underline{A}$$

Volume Flux

- Volume flux (or flow rate),

$$Q = \int_A \underline{V} \cdot d\underline{A}$$

  - Average velocity

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int_A \underline{V} \cdot d\underline{A}$$

  - If $\underline{V}$ = constant

$$Q = \underline{V} \cdot \underline{A}$$

  - If $\underline{V}$ is normal to $\underline{A}$

$$Q = VA \quad \left(V = \left|\underline{V}\right|\right)$$

Momentum Flux

- The rate of momentum carried by fluid flow through an area A:

$$\underline{\dot{\mathrm{P}}} = \int_A \underline{V} d\dot{m} = \int_A \underline{V}(\rho \underline{V} \cdot d\underline{A})$$

If ρ = constant and $\underline{V}$ = constant,

$$\underline{\dot{\mathrm{P}}} = \underline{V} \int_A \rho \underline{V} \cdot d\underline{A} = (\rho \underline{V} \cdot \underline{A})\underline{V} = \dot{m}\underline{V}$$

or, in components

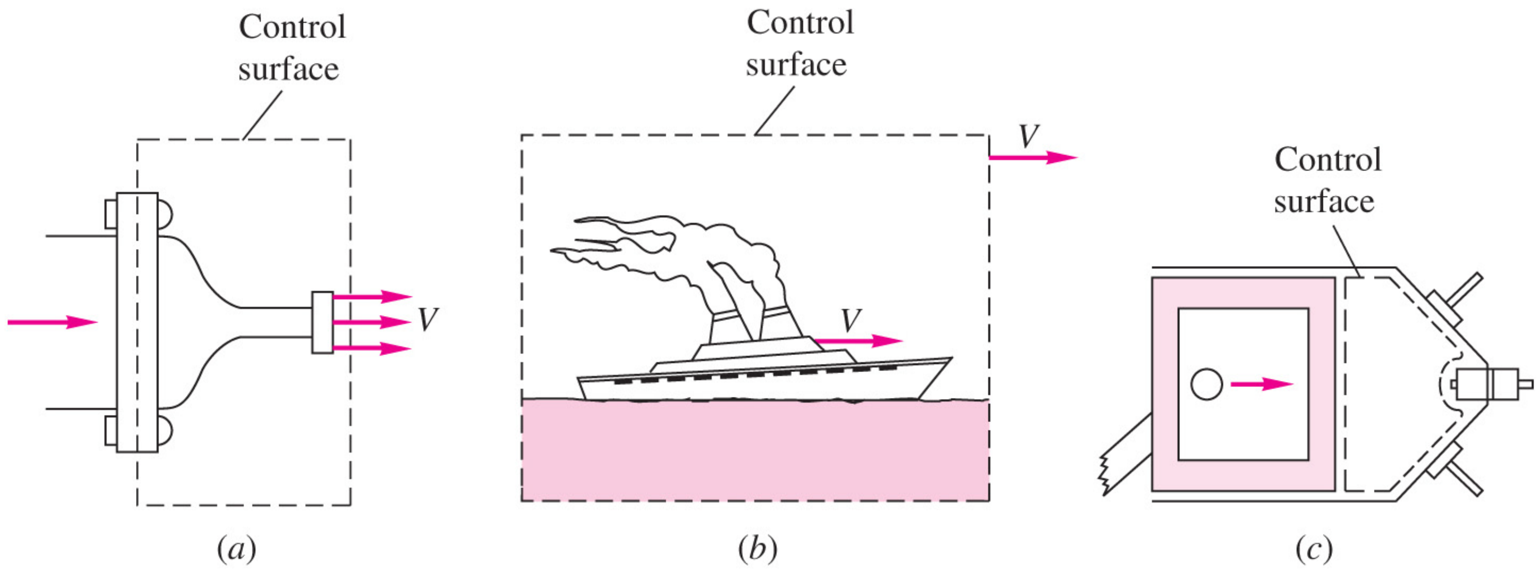
$$\dot{\mathrm{P}}_x = \dot{m}u$$
$$\dot{\mathrm{P}}_y = \dot{m}v$$
$$\dot{\mathrm{P}}_z = \dot{m}w$$

where, $\underline{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$

System vs. Control volume

- **System**: A collection of (real) matter of fixed identity.

- **Control volume**: A (geometric or an imaginary) volume in space through which fluid may flow, which may move or deform.



Reynolds Transport Theorem (RTT)

- General RTT for a moving and deforming CV,

$$\underbrace{\frac{DB_{sys}}{Dt}}_{\substack{\text{time rate of change}\\ \text{of } B \text{ for a system}}} = \underbrace{\frac{D}{Dt}\int_{CV(\underline{x},t)} \beta\rho d\forall}_{\substack{\text{time rate of change}\\ \text{of } B \text{ in } CV}} + \underbrace{\int_{CS(\underline{x},t)} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net flux of } B\\ \text{across } CS}}$$

- For a fixed CV,

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \beta\rho d\forall + \int_{CS} \beta\rho \underline{V} \cdot d\underline{A}$$

Simplified form for a steady flow with discrete CS's of uniform flow

$$\frac{DB_{sys}}{Dt} = \sum(\beta\dot{m})_{\text{out}} - \sum(\beta\dot{m})_{\text{in}}$$

Continuity Eq.

- RTT with $B = m$ and $\beta = 1$,

$$\frac{Dm_{sys}}{Dt} = 0 = \frac{D}{Dt}\int_{CV} \rho d\mathcal{V} + \int_{CS} \rho \underline{V_R} \cdot d\underline{A}$$

- Simplified form:

$$0 = \sum \dot{m}_{\text{out}} - \sum \dot{m}_{\text{in}}$$

- Conduit flow with one inlet (1) and one outlet (2):

$$\rho_2 V_2 A_2 - \rho_1 V_1 A_1 = 0$$

If ρ = constant

$$V_1 A_1 = V_2 A_2$$

Momentum Equations

- RTT with $B = m\underline{V}$ and $\beta = \underline{V}$,

$$\frac{D\left(m\underline{V}\right)_{sys}}{Dt} = \sum \underline{F} = \frac{D}{Dt}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho \underline{V}_R \cdot d\underline{A}$$

- Simplified form:

$$\sum \underline{F} = \sum \left(\dot{m}\underline{V}\right)_{\text{out}} - \sum \left(\dot{m}\underline{V}\right)_{\text{in}}$$

Note:

$$\frac{D\left(m\underline{V}\right)}{Dt} = \frac{Dm}{Dt}\underline{V} + m\frac{D\underline{V}}{Dt}$$

$$= m\underline{a} \quad \left(\because \frac{Dm}{Dt} = 0\right)$$

or in component form,

$$\sum F_x = \sum (\dot{m}u)_{\text{out}} - \sum (\dot{m}u)_{\text{in}}$$

$$\sum F_y = \sum (\dot{m}v)_{\text{out}} - \sum (\dot{m}v)_{\text{in}}$$

$$\sum F_z = \sum (\dot{m}w)_{\text{out}} - \sum (\dot{m}w)_{\text{in}}$$

Momentum Equations – Contd.

- External forces:

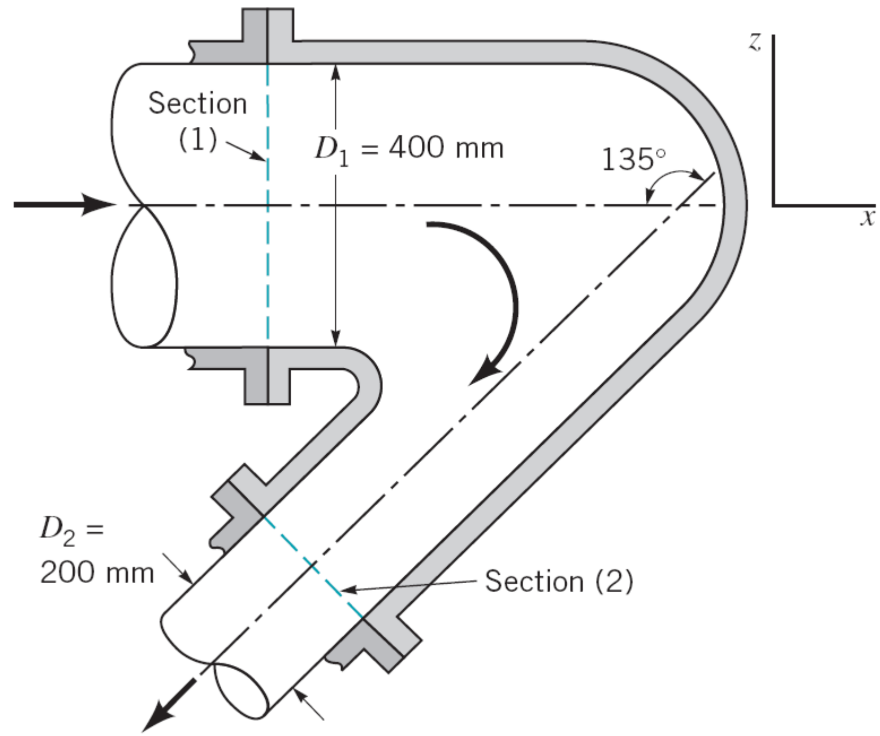
$$\sum \underline{F} = \underline{F}_{\text{Surface}} + \underline{F}_{\text{Body}} + \underline{F}_{\text{Other forces}}$$

  - $\underline{F}_{\text{Surface}}$ = Pressure or shearing forces
  - $\underline{F}_{\text{Body}}$ = Gravity forces (i.e., weight)
  - $\underline{F}_{\text{other forces}}$ = Anchoring forces or reaction forces

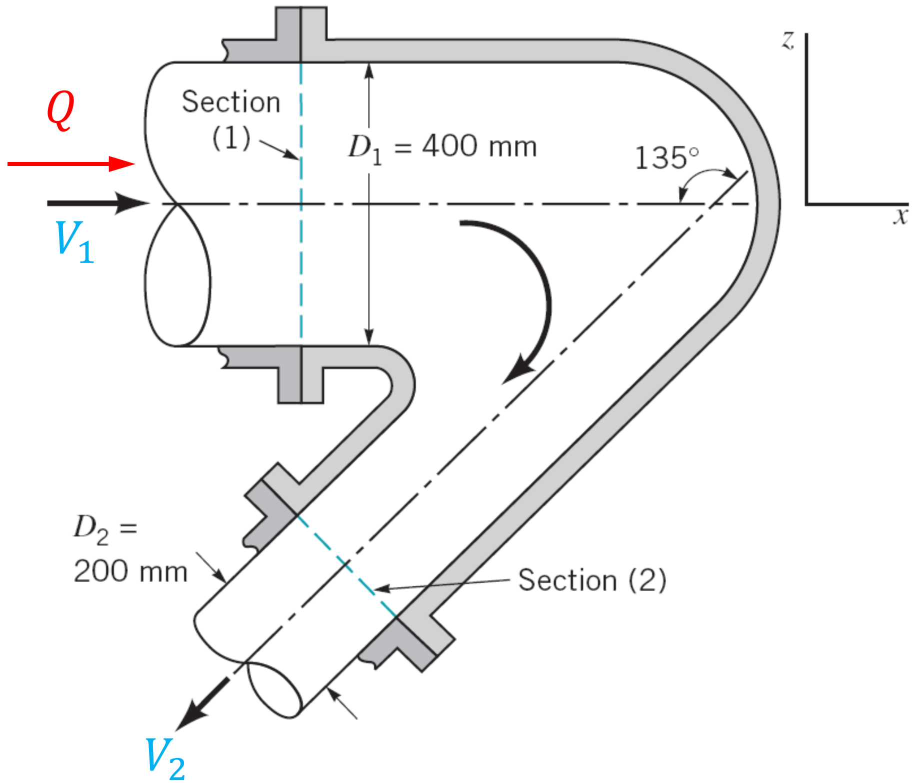
Note: Shearing forces can be avoided by carefully selecting the CV such that the force component is not exposed on the CS

Example (Bend)

5.34 A converging elbow (see Fig. P5.34) turns water through an angle of 135° in a vertical plane. The flow cross section diameter is 400 mm at the elbow inlet, section (1), and 200 mm at the elbow outlet, section (2). The elbow flow passage volume is 0.2 m^3 between sections (1) and (2). The water volume flowrate is 0.4 m^3/s and the elbow inlet and outlet pressures are 150 kPa and 90 kPa. The elbow mass is 12 kg. Calculate the horizontal (x direction) and vertical (z direction) anchoring forces required to hold the elbow in place.



Example (Bend) – Contd.

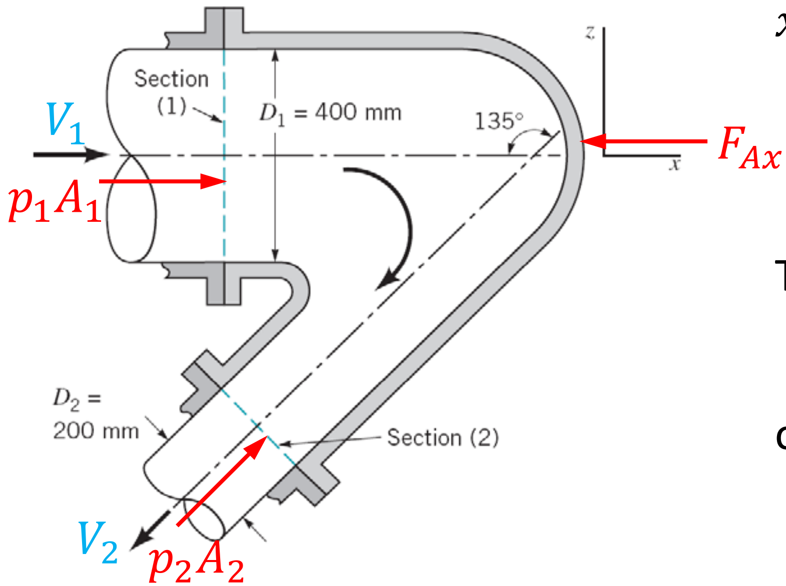


$$Q = 0.4 \ \text{m}^3/\text{s}$$
$$D_1 = 0.4 \ \text{m}$$
$$D_2 = 0.2 \ \text{m}$$

$$V_1 = \frac{Q}{A_1} = \frac{Q}{\pi D_1^2/4} = \frac{0.4}{\pi (0.4)^2/4} = 3.18 \ \text{m/s}$$

$$V_2 = \frac{Q}{A_2} = \frac{Q}{\pi D_2^2/4} = \frac{0.4}{\pi (0.2)^2/4} = 12.73 \ \text{m/s}$$

Example (Bend) – Contd.



x-momentum:

$$\sum F_x = (\dot{m}u)_{\text{out}} - (\dot{m}u)_{\text{in}}$$

Thus,

$$-F_{Ax} + p_1A_1 + p_2A_2\cos 45^\circ = (\rho Q)(-V_2\cos 45^\circ) - (\rho Q)(V_1)$$

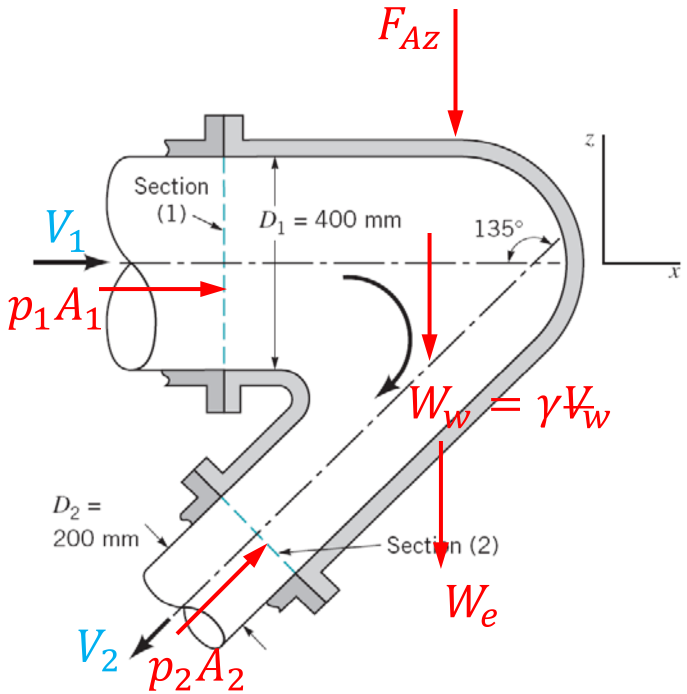
or

$$F_{Ax} = p_1A_1 + p_2A_2\cos 45^\circ + (\rho Q)(V_1 + V_2\cos 45^\circ)$$

$$= (150{,}000)\frac{\pi(0.4)^2}{4} + (50{,}000)\frac{\pi(0.2)^2}{4}\cos 45^\circ + (999)(4)(3.18 + 12.73\cos 45^\circ)$$

$$\therefore F_{Ax} = 25{,}700\ \text{N}$$

Example (Bend) – Contd.



z-momentum:

$$\sum F_z = (\dot{m}w)_{\text{out}} - (\dot{m}w)_{\text{in}}$$

Thus,

$$-F_{Az} + p_2 A_2 \sin 45^\circ - W_w - W_e = (\rho Q)(-V_2 \sin 45^\circ) - (\rho Q)(0)$$

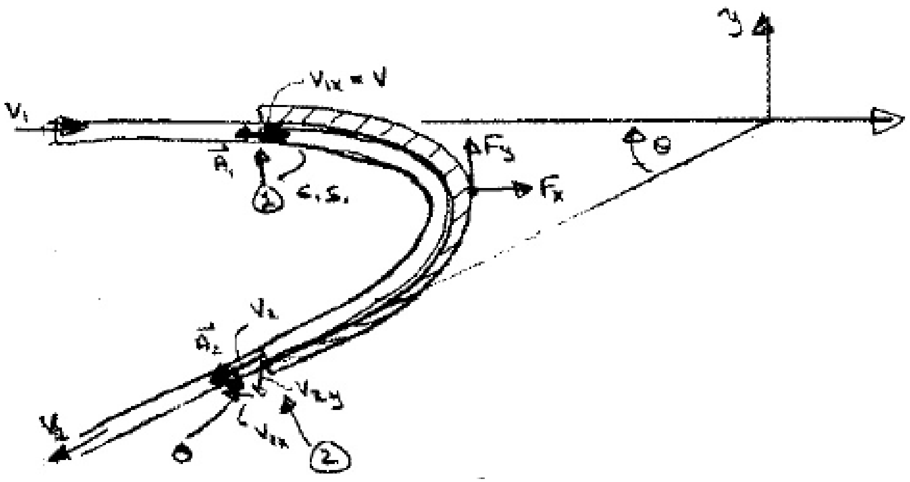
or

$$F_{Az} = p_2 A_2 \cos 45^\circ - \gamma \forall_w - W_e + (\rho Q)(V_2 \sin 45^\circ)$$

$$= (50{,}000)\frac{\pi (0.2)^2}{4} \sin 45^\circ - (9800)(0.2) - (12)(9.81) + (999)(4)(12.73 \sin 45^\circ)$$

$$\therefore F_{Ax} = 8{,}920 \text{ N}$$

Typical Example (1): Vane



Energy eq.:

$$p_1 + \frac{1}{2}\rho V_1^2 + z_1 = p_2 + \frac{1}{2}\rho V_2^2 + z_2 + h_L$$

with $p_1 = p_2 = 0$, $z_1 \approx z_2$, and $h_L \approx 0$,

$$\therefore V_1 = V_2 = V_j$$

Continuity:

$$V_1 A_1 = V_2 A_2 = V_j A_j \quad \Rightarrow \quad \dot{m} = \rho V_j A_j$$

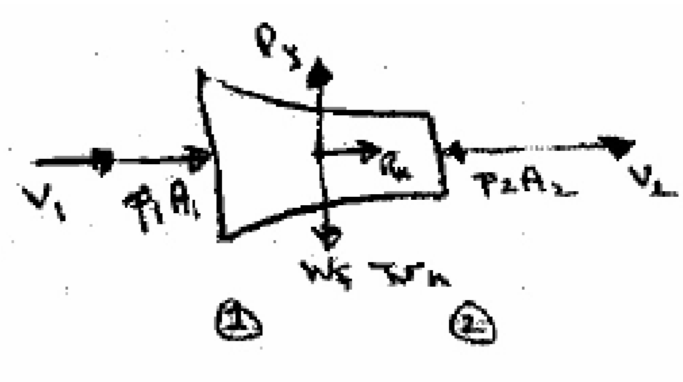
x-momentum:

$$F_x = \underbrace{\dot{m}(-V_2 \cos\theta)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$F_y - W_{\text{fluid}} - W_{\text{vane}} = \underbrace{\dot{m}(-V_2 \sin\theta)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Typical Example (2): Nozzle



Continuity:

$$V_1 A_1 = V_2 A_2$$

$$\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$$

Energy eq. with $p_2 = 0$ and $z_1 = z_2$:

$$p_1 + \frac{1}{2}\rho V_1^2 = \frac{1}{2}\rho V_2^2 + h_L$$

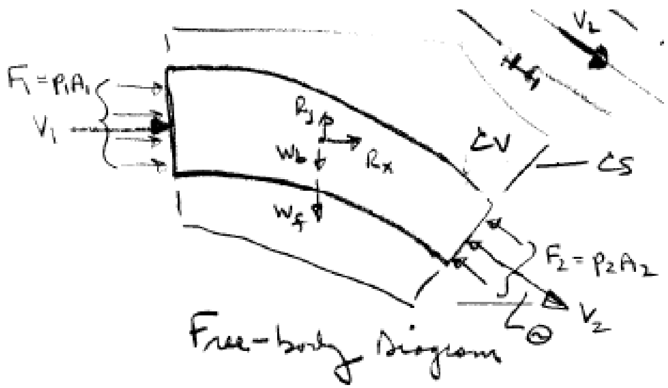
x-momentum:

$$R_x + p_1 A_1 = \underbrace{\dot{m}(V_2)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$R_y - W_{\text{fluid}} - W_{\text{nozzle}} = \underbrace{\dot{m}(0)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Typical Example (3): Bend



Continuity:

$$V_1 A_1 = V_2 A_2$$

$$\dot{m} = \rho V_1 A_1 = \rho V_2 A_2$$

Energy eq.:

$$p_1 + \frac{1}{2}\rho V_1^2 + z_1 = p_2 + \frac{1}{2}\rho V_2^2 + z_2 + h_L$$

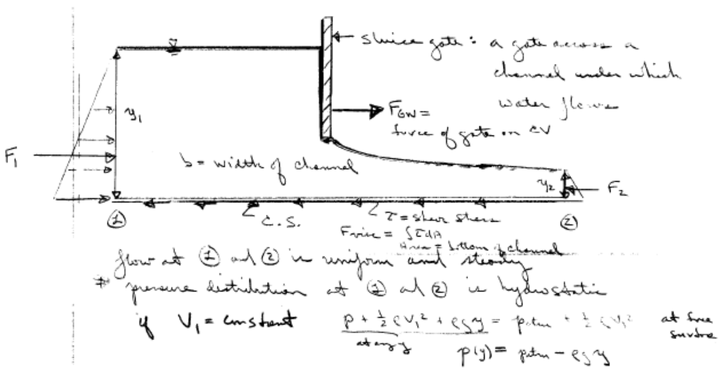
x-momentum:

$$R_x + p_1 A_1 - p_2 A_2 \cos\theta = \underbrace{\dot{m}(V_2 \cos\theta)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$R_y + p_2 A_2 \sin\theta - W_{\text{fluid}} - W_{\text{bend}} = \underbrace{\dot{m}(-V_2 \sin\theta)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Typical Example (4): Sluice gate



Continuity:

$$V_1(y_1 b) = V_2(y_2 b)$$

$$\dot{m} = \rho V_1(y_1 b) = \rho V_2(y_2 b)$$

Energy (Bernoulli) eq. with $p_1 = p_2$ and $h_L \approx 0$:

$$\frac{1}{2}\rho V_1^2 + y_1 = \frac{1}{2}\rho V_2^2 + y_2$$

x-momentum:

$$F_{GW} + \underbrace{\gamma\left(\frac{y_1}{2}\right)(y_1 b)}_{\bar{p}_1 A_1} - \underbrace{\gamma\left(\frac{y_2}{2}\right)(y_2 b)}_{\bar{p}_2 A_2} = \underbrace{\dot{m}(V_2)}_{\text{out}} - \underbrace{\dot{m}(V_1)}_{\text{in}}$$

y-momentum:

$$0 = \underbrace{\dot{m}(0)}_{\text{out}} - \underbrace{\dot{m}(0)}_{\text{in}}$$

Energy Equation

- RTT with $B = E$ and $\beta = e$,

$$\dot{Q} - \dot{W} = \frac{DE_{sys}}{Dt} = \frac{D}{Dt}\int_{CV} e\rho dV + \int_{CS} e\rho \underline{V_R} \cdot d\underline{A} = 0$$

- Simplified form:

$$\frac{p_{in}}{\gamma} + \alpha_{in}\frac{V_{in}^2}{2g} + z_{in} + h_p = \frac{p_{out}}{\gamma} + \alpha_{out}\frac{V_{out}^2}{2g} + z_{out} + h_t + h_L$$

- α : kinetic energy correction factor ($\alpha = 1$ for uniform flow across CS)
- V in energy equation refers to average velocity $\bar{V}$

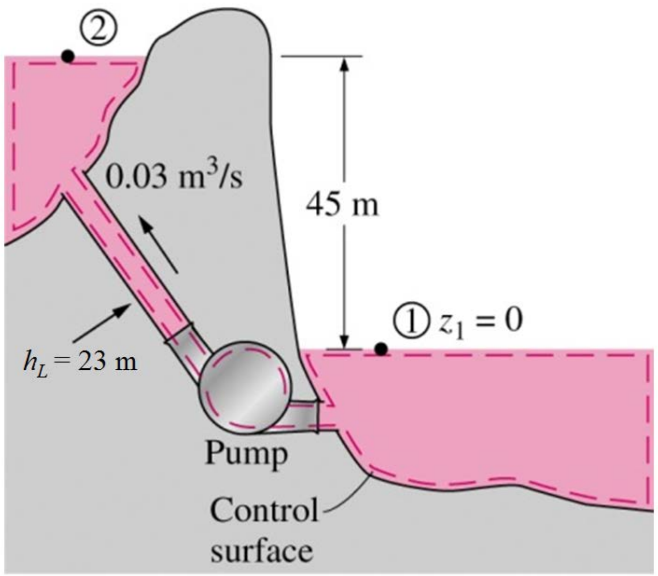
Simplified Energy Equation

Uniform flow across CS's:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_1 + h_t + h_L$$

- Pump head $h_p = \dot{W}_p/\dot{m}g = \dot{W}_p/\rho Q g = \dot{W}_p/\gamma Q$
- Turbine head $h_t = \dot{W}_t/\dot{m}g = \dot{W}_t/\rho Q g = \dot{W}_t/\gamma Q$
- Head loss $h_L = \text{loss}/g = (\hat{u}_2 - \hat{u}_1)/g - \dot{Q}/\dot{m}g > 0$

Example (Pump)



Energy equation:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

With $p_1 = p_2 = 0$, $V_1 = V_2 \approx 0$, $h_t = 0$, and $h_L = 23\ \mathrm{m}$

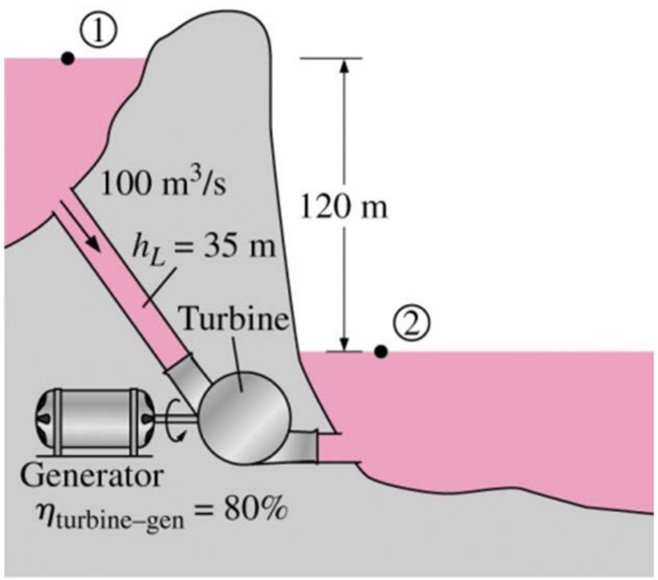
$$h_p = (z_2 - z_1) + h_L = 45 + 23 = 68\ \mathrm{m}$$

Pump power,

$$\dot{W}_p = h_p \gamma Q = \frac{(68)(9790)(0.03)}{746} = 80\ \mathrm{hp}$$

Note: 1 hp = 746 N·m/s(= W) or 550 ft·lbf/s

Example (Turbine)



Energy equation:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + h_t + h_L$$

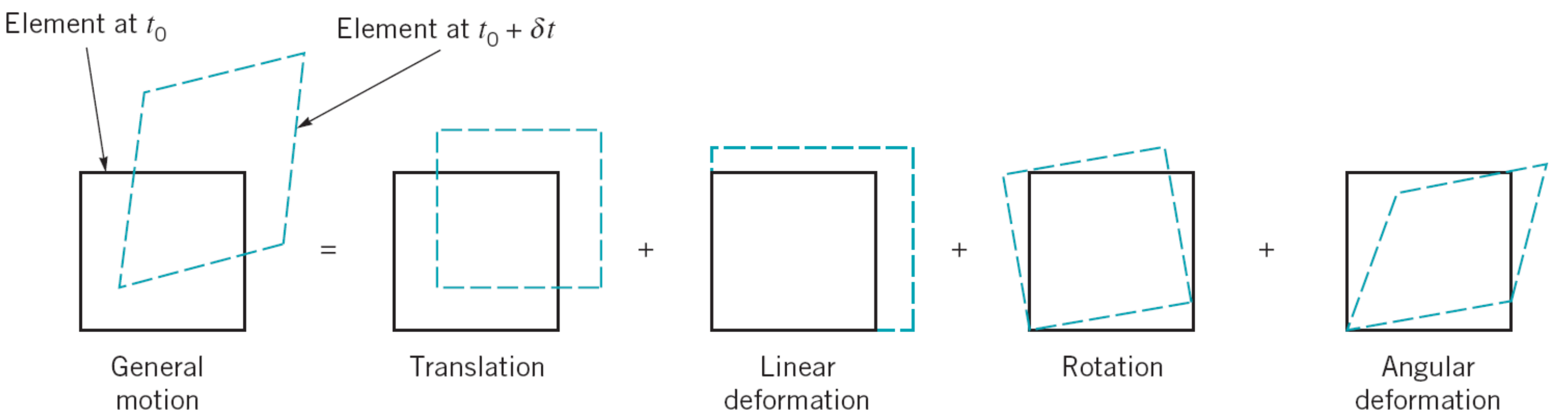
With $p_1 = p_2 = 0$, $V_1 = V_2 \approx 0$, $h_p = 0$, and $h_L = 35\ \mathrm{m}$

$$h_t = (z_1 - z_2) - h_L = 120 - 35 = 85\ \mathrm{m}$$

Pump power,

$$\dot{W}_t = h_t \gamma Q = (85)(9790)(100) = 83.2\ \mathrm{MW}$$

Fluid Element Kinematics



- Linear deformation(dilatation): $\nabla \cdot \underline{V}$

  $\Rightarrow$ if the fluid is **incompressible** $\boldsymbol{\nabla} \cdot \underline{\boldsymbol{V}} = \mathbf{0}$

- Rotation(vorticity): $\underline{\xi} = 2\underline{\omega} = \nabla \times \underline{V}$

  $\Rightarrow$ if the fluid is **irrotational** $\boldsymbol{\nabla} \times \underline{\boldsymbol{V}} = \mathbf{0}$

- Angular deformation is related to shearing stress

  (e.g., $\tau_{ij} = 2\mu\varepsilon_{ij}$ for Newtonian fluids)

Mass Conservation

For a fluid particle,

$$\lim_{CV\to 0}\left[\int_{CV}\frac{\partial\rho}{\partial t}d\forall+\int_{CS}\rho\underline{V}\cdot d\underline{A}\right]$$

$$=\lim_{CV\to 0}\int_{CV}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)\right]d\forall=0$$

$$\therefore\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)=0$$

For an incompressible flow: $\nabla\cdot\underline{V}=0$

Momentum Conservation

$$\lim_{CV\to 0}\left[\int_{CV}\frac{\partial \underline{V}}{\partial t}\rho d\forall + \int_{CS}\underline{V}\rho\underline{V}\cdot\underline{dA}\right] = \sum \underline{F}$$

or

$$\lim_{CV\to 0}\int_{CV}\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right)d\forall = \sum \underline{F}$$

$$\therefore \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \sum \underline{f} \qquad (\underline{f} = \underline{F}\text{ per unit volume})$$

$$\Rightarrow \underbrace{\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right)}_{=\frac{D\underline{V}}{Dt}=\underline{a}} = \underbrace{-\rho g\hat{k}}_{\text{body force due to gravity force}} \underbrace{\underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\nabla\cdot\tau_{ij}}_{\text{viscous shear force}}}_{\text{surface force}}$$

Navier-Stokes Equations

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Cylindrical Coordinates

Continuity:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Momentum:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

Exact Solutions of NS Eqns.

The flow of interest is assumed additionally (than incompressible & Newtonian) as:

1) Steady (i.e., $\boldsymbol{\partial/\partial t = 0}$ for any variable)

2) Parallel such that the y-component of velocity is zero (i.e., $\boldsymbol{v = 0}$)

3) Fully developed (i.e., $\boldsymbol{\partial u/\partial x}$ from the continuity equation and the assumptions 2 and 3)

4) Purely two dimensional (i.e., $\boldsymbol{w = 0}$ and $\boldsymbol{\partial/\partial z = 0}$ for any velocity component)

Boundary Conditions

Common BC's:

- **No-slip condition** ($\underline{V}_{fluid} = \underline{V}_{wall}$; for a stationary wall $\underline{V}_{wall} = 0$)
- Interface boundary condition ($\underline{V}_A = \underline{V}_B$ and $\tau_{s,A} = \tau_{s,B}$)
- Free-surface boundary condition ($p_{liquid} = p_{gas}$ and $\tau_{s,fluid} = 0$)

Other BC's:

- Inlet/outlet boundary condition
- Symmetry boundary condition
- Initial condition (for unsteady flow problem)

Solving the NS Eqns

1) Set up the problem and geometry (e.g., sketches), identifying all relevant dimensions and parameters.

2) List all appropriate assumptions, approximations, simplifications, and boundary conditions.

3) Simplify the differential equations of motion (continuity and Navier-Stokes) as much as possible.

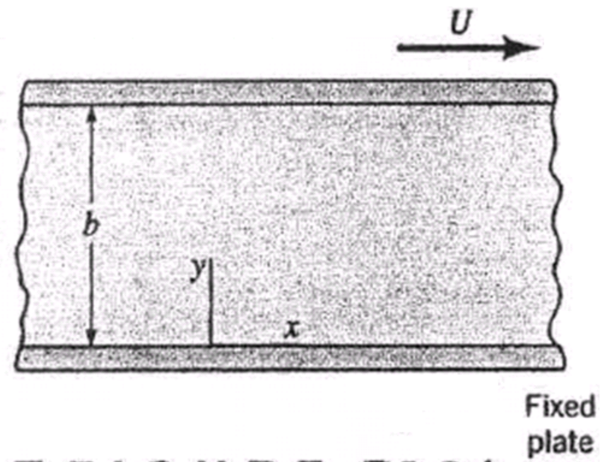
4) Integrate the equations, leading to one or more constants of integration

5) Apply boundary conditions to solve for the constants of integration.

6) Verify your results.

Example: Couette Flow

6.96 The viscous, incompressible flow between the parallel plates shown in Fig. P6.96 is caused by both the motion of the top plate and a pressure gradient, $\partial p/\partial x$. As noted in Section 6.9.2, an important dimensionless parameter for this type of problem is $P = -(b^2/2\,\mu U)\,(\partial p/\partial x)$ where μ is the fluid viscosity. Make a plot of the dimensionless velocity distribution (similar to that shown in Fig. 6.32*b*) for $P = 3$. For this case where does the maximum velocity occur?



■ FIGURE P6.96

Example: Couette Flow

Assumptions **1** The plates are infinite in *x* and *z*. **2** The flow is steady, i.e., $\partial/\partial t$ of anything is zero. **3** This is a parallel flow (we assume the *y*-component of velocity, *v*, is zero). **4** The fluid is incompressible and Newtonian with constant properties, and the flow is laminar. **5** A constant pressure gradient is applied in the *x*-direction such that pressure changes linearly with respect to *x*

6 The velocity field is purely two-dimensional, meaning here that $w = 0$ and $\partial/\partial z$ of any velocity component is zero. **7** Gravity acts in the negative *z*-direction (into the page in Fig. 9–55). We express this mathematically as $\vec{g} = -g\vec{k}$, or $g_x = g_y = 0$ and $g_z = -g$.

$$\frac{\partial u}{\partial x} + \underbrace{\frac{\partial v}{\partial y}}_{3} + \underbrace{\frac{\partial w}{\partial z}}_{6} = 0$$

$$\rho\left[\underbrace{\frac{\partial u}{\partial t}}_{2} + \underbrace{u\frac{\partial u}{\partial x}}_{\text{Continuity}} + \underbrace{v\frac{\partial u}{\partial y}}_{3} + \underbrace{w\frac{\partial u}{\partial z}}_{6}\right] = -\frac{\partial p}{\partial x} + \underbrace{\rho g_x}_{7} + \mu\left[\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{Continuity}} + \frac{\partial^2 u}{\partial y^2} + \underbrace{\frac{\partial^2 u}{\partial z^2}}_{6}\right]$$

Example: Couette Flow

Simplified NS-equation:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2}$$

or

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)$$

Integrate twice, noting that $\partial p/\partial x$ is a constant,

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants of integration.

Example: Couette Flow

Boundary condition (1): $u = 0$ at $y = 0$,

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) \times 0 + C_1 \times 0 + C_2 = 0 \quad \Rightarrow \quad C_2 = 0$$

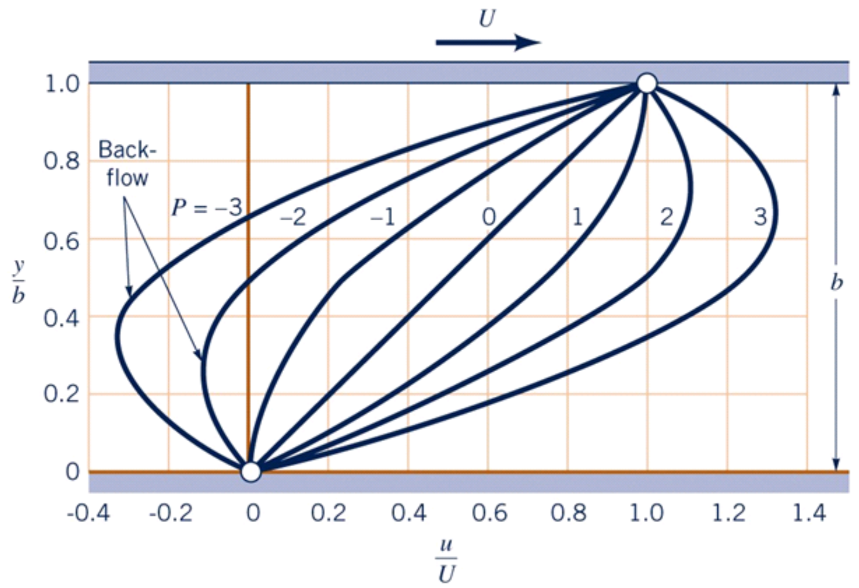
Boundary condition (2): $u = U$ at $y = h$,

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) b^2 + C_1 b + 0 = U \quad \Rightarrow \quad C_1 = \frac{U}{b} - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right) b$$

Finally,

$$u = \frac{U}{b} y + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by)$$

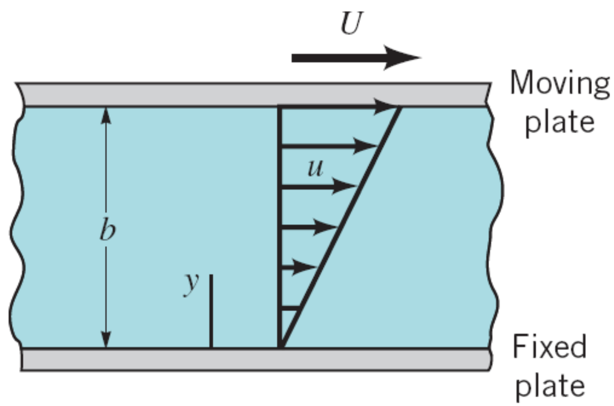
Example: Couette Flow



Let: $P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$

$$\frac{u}{U} = \frac{y}{b} + P\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

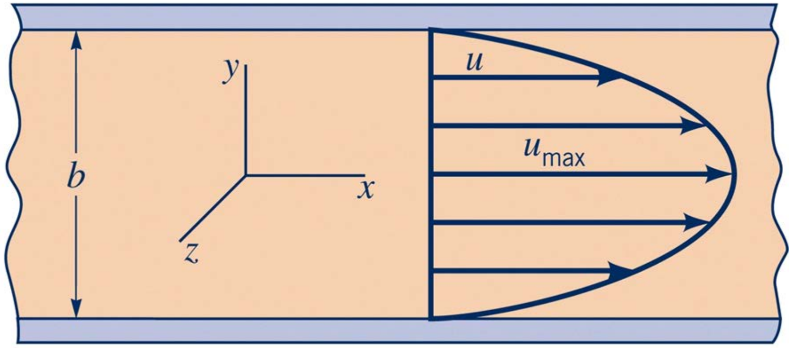
Special Case(1): Without $\partial p/\partial x$



$$u(y) = \frac{U}{b} y$$

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0} = \frac{\mu U}{b}$$

Special Case(2): Both plates fixed



B.C.: $u(0) = u(b) = 0$

$$\therefore u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by)$$

$$q = \int_{-h}^{h} u dy = -\frac{b^3}{12\mu}\left(\frac{\partial p}{\partial x}\right); \quad \bar{V} = \frac{q}{b} = -\frac{b^2}{12\mu}\left(\frac{\partial p}{\partial x}\right);$$

$$u_{\max} = u\left(\frac{b}{2}\right) = -\frac{b^2}{8\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{3}{2}\bar{V};$$

$$\tau_w = \mu \left.\frac{du}{dy}\right)_{y=0} = -\frac{b}{2}\left(\frac{\partial p}{\partial x}\right)$$

Buckingham Pi Theorem

- For any physically meaningful equation involving $\boldsymbol{n}$ **variables**, such as

$$u_1 = f(u_2, u_3, \cdots, u_n)$$

- With minimum number of $\boldsymbol{m}$ **reference dimensions**, the equation can be rearranged into product of $\boldsymbol{r}$ **dimensionless pi terms**.

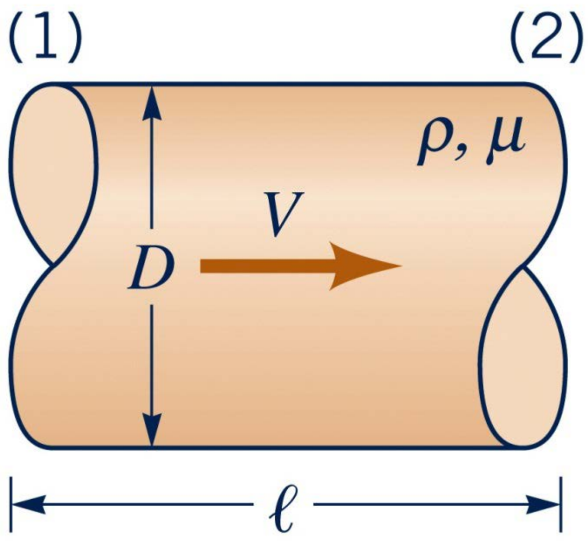
$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_r)$$

where,

$$\boldsymbol{r = n - m}$$

Repeating Variable Method

Example: The pressure drop per unit length Δp_ℓ in a pipe flow is a function of the pipe diameter D and the fluid density ρ, viscosity μ, and velocity V.



$$\Delta p_\ell = (p_1 - p_2)/\ell$$

Repeating Variable Method – Contd.

Step 1: List all variables that are involved in the problem

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Step 2: Express each of the variables in terms of basic dimensions (either MLT or FLT system)

Δp_ℓ	D	ρ	μ	V
$\{ML^{-2}T^{-2}\}$	$\{L\}$	$\{ML^{-3}\}$	$\{ML^{-1}T^{-1}\}$	$\{LT^{-1}\}$

Step 3: Determine the required number of pi terms

$$r = n - m = 5 - 3 = 2$$

Step 4: Select m = 3 repeating variables

D (for L), V (for T), and ρ (for M)

Repeating Variable Method – Contd.

Step 5: Form a pi term for one of the non-repeating variables

$$\Pi_1 = D^a V^b \rho^c \Delta p_\ell \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-2}T^{-2}) \doteq M^0 L^0 T^0$$

$$\therefore \Pi_1 = D^{-1} V^{-2} \rho^{-1} \Delta p_\ell = \frac{\Delta p_\ell D}{\rho V^2}$$

Step 6: Repeat step 5 for each of the remaining non-repeating variables

$$\Pi_2 = D^a V^b \rho^c \mu \doteq (L)^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1}) \doteq M^0 L^0 T^0$$

$$\therefore \Pi_2 = D^{-1} V^{-1} \rho^{-1} \mu = \frac{\mu}{DV\rho}$$

Repeating Variable Method – Contd.

Step 7: Check all the resulting pi terms to make sure they are dimensionless and independent

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq F^0 L^0 T^0; \quad \Pi_2 = \frac{\mu}{DV\rho} \doteq F^0 L^0 T^0$$

Step 8: Express the final form as a relationship among the pi terms

$$\Pi_1 = \phi(\Pi_2)$$

or

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

For,

$$\Pi_1 = \phi(\Pi_2, \ldots, \Pi_n)$$

Similarity requirements:

$$\Pi_{2,\mathrm{model}} = \Pi_{2,\mathrm{prototype}}$$
$$\vdots$$
$$\Pi_{n,\mathrm{model}} = \Pi_{n,\mathrm{prototype}}$$

Prediction equation:

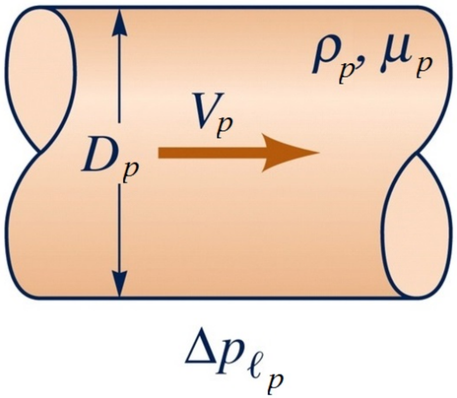
$$\Pi_{1,\mathrm{model}} = \Pi_{1,\mathrm{prototype}}$$

Example: Model Testing

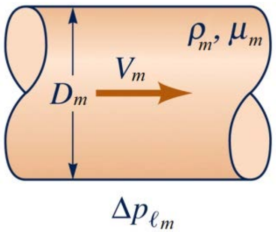
Dimensional:

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Prototype



Model



D_p = 1 m
ρ_p = 1.23 kg/m^3
μ_p = 1.79 × 10^{-5} N·s/m^2
V_p = 10 m/s

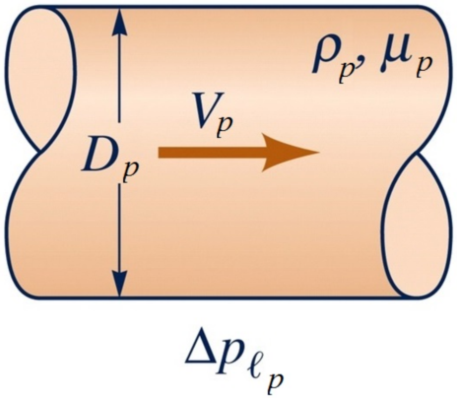
D_m = 0.1 m
ρ_m = 998 kg/m^3
μ_m = 1.12 × 10^{-3} N·s/m^2
V_m = ?

Example: Model Testing

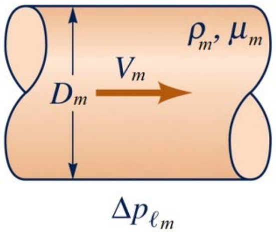
Non-dimensional:

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

<u>Prototype</u>



<u>Model</u>



$$\Pi_{1,p} = \frac{\Delta p_{\ell_p} D_p}{\rho_p V_p^2}$$

$$\Pi_{2,p} = \frac{\rho_p V_p D_p}{\mu_p}$$

$$\Pi_{1,m} = \frac{\Delta p_{\ell_m} D_m}{\rho_m V_m^2}$$

$$\Pi_{2,m} = \frac{\rho_m V_m D_m}{\mu_m}$$

Example: Model Testing

- Model design condition (similarity requirements)

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho_p V_p D_p}{\mu_p}$$

or

$$V_m = \left(\frac{\rho_p}{\rho_m}\right)\left(\frac{\mu_m}{\mu_p}\right)\left(\frac{D_p}{D_m}\right)V_p$$

$$= \left(\frac{1.23}{998}\right)\left(\frac{1.12 \times 10^{-3}}{1.79 \times 10^{-5}}\right)\left(\frac{1}{0.1}\right)(10)$$

$$\therefore V_m = 7.71\ \text{m/s}$$

Example: Model Testing

If Δp_{ℓ_m} is measured at 27.6 Pa/m from the model test, then Δp_{ℓ_p} = ?

- Prediction equation

$$\frac{\Delta p_{\ell_p} D_p}{\rho_p V_p^2} = \frac{\Delta p_{\ell_m} D_m}{\rho_m V_m^2}$$

or

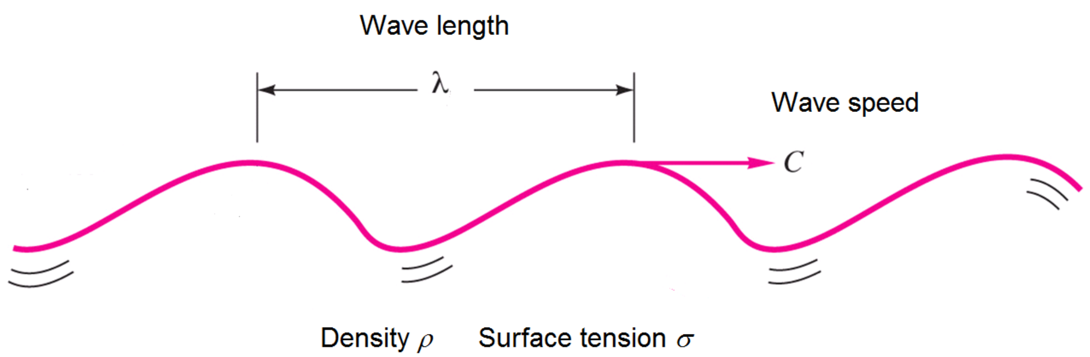
$$\Delta p_{\ell_p} = \left(\frac{D_m}{D_p}\right)\left(\frac{\rho_p}{\rho_m}\right)\left(\frac{V_p}{V_m}\right)^2 \Delta p_{\ell_m}$$

$$= \left(\frac{0.1}{1}\right)\left(\frac{1.23}{998}\right)\left(\frac{10}{7.71}\right)^2 (27.6)$$

$$\therefore \Delta p_{\ell_p} = 5.72 \times 10^{-3} \ \mathrm{Pa/m}$$

Example: Problems with one Pi

The speed of propagation C of a capillary wave in deep water is known to be a function only of density ρ, wavelength λ, and surface tension σ. Find the proper functional relationship, completing it with a dimensionless constant. For a given density and wavelength, how does the propagation speed change if surface tension is doubled?



Example: Problems with one Pi

$$C = f(\rho, \lambda, \sigma)$$

C	ρ	λ	σ
$\{LT^{-1}\}$	$\{ML^{-3}\}$	$\{L\}$	$\{MT^{-2}\}$

$$r = n - m = 5 - 3 = 1$$

$$\Pi = \rho^a \lambda^b \sigma^c C \doteq (ML^{-3})^a (L)^b (MT^{-2})^c (LT^{-1}) \doteq M^{(a+c)} L^{(-3a+b+1)} T^{(-2c\,-1)} \doteq M^0 L^0 T^0$$

$$a = \frac{1}{2}, \qquad b = \frac{1}{2}, \qquad c = -\frac{1}{2}$$

$$\Pi = \rho^{\frac{1}{2}} \lambda^{\frac{1}{2}} \sigma^{-\frac{1}{2}} C$$

Example: Problems with one Pi

$$\therefore \Pi = C\sqrt{\frac{\rho\lambda}{\sigma}} = \text{Constant}, K$$

Let C' be the wave speed when surface tension σ is doubled,

$$C\sqrt{\frac{\rho\lambda}{\sigma}} = K \quad \text{and} \quad C'\sqrt{\frac{\rho\lambda}{(2\sigma)}} = K$$

or

$$C\sqrt{\frac{\rho\lambda}{\sigma}} = C'\sqrt{\frac{\rho\lambda}{(2\sigma)}}$$

or

$$C' = C\sqrt{\frac{\rho\lambda}{\sigma}} \times \sqrt{\frac{(2\sigma)}{\rho\lambda}}$$

$$\therefore C' = \sqrt{2}C$$

Example: Problems with one Pi

Alternatively,

$$C = K\sqrt{\frac{\sigma}{\rho\lambda}}$$

Then,

$$C' = K\sqrt{\frac{(2\sigma)}{\rho\lambda}} = \sqrt{2}\cdot K\sqrt{\frac{\sigma}{\rho\lambda}} = \sqrt{2}\cdot C$$

Thus, C increases as $\sqrt{2}$, or about 41 percent.