Review for Exam2

11. 15. 2013

Hyunse Yoon, Ph.D.
Assistant Research Scientist
IIHR-Hydroscience & Engineering
University of Iowa

Reynolds Transport Theorem (RTT)

For any extensive property B,

$$\underbrace{\frac{DB_{sys}}{Dt}}_{\substack{\text{time rate of change}\\ \text{of } B \text{ for a system}}} = \underbrace{\frac{d}{dt}\int_{CV} \beta\rho d\forall}_{\substack{\text{time rate of change}\\ \text{of } B \text{ in } CV}} + \underbrace{\int_{CS} \beta\rho \underline{V}_R \cdot d\underline{A}}_{\substack{\text{net flux of } B\\ \text{across } CS}}$$

where,

- $B = (m, m\underline{V}, E)$; $\beta = B/m = (1, \underline{V}, e)$ for mass, momentum and energy conservation laws, respectively
- $\underline{V}_R = \underline{V} - \underline{V}_S$; $\underline{V}$ = fluid velocity; $\underline{V}_S = CS$ velocity

Continuity Equation

For $B = m$ thus $\beta = 1$,

$$\frac{Dm_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$$

Special cases:

- Incompressible fluid (ρ =constant) $\int_{CS} \underline{V} \cdot d\underline{A} = -\frac{\partial}{\partial t}\int_{CV} d\forall$
- Steady flow $\int_{CS} \rho \underline{V} \cdot d\underline{A} = 0$
- $\underline{V}$ = constant over discrete $d\underline{A}$ $\int_{CS} \rho \underline{V} \cdot d\underline{A} = \sum_{CS} \rho \underline{V} \cdot \underline{A}$
- Steady 1D flow in a conduit $\sum_{CS} \rho \underline{V} \cdot \underline{A} = 0 \Rightarrow \rho_1 V_1 A_1 = \rho_2 V_2 A_2$

$\Rightarrow$ if ρ = constant, $V_1 A_1 = V_2 A_2$ or $Q_1 = Q_2$

Useful Definitions

- Mass flux (mass flow rate) $\dot{m} = \int_A \rho \underline{V} \cdot d\underline{A}$
- Volume flux (flow rate) $Q = \int_A \underline{V} \cdot d\underline{A}$
- Average velocity $\bar{V} = Q/A$

Note:

- If $\underline{V}$ = constant $Q = \underline{V} \cdot \underline{A}$
- If ρ = constant $\dot{m} = \rho Q$

Momentum Equation

For $B = m\underline{V}$ thus $\beta = \underline{V}$,

$$\sum \underline{F} = \frac{D\left(m\underline{V}\right)_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \underline{V}\rho d\forall + \int_{CS} \underline{V}\rho\underline{V}\cdot\underline{dA}$$

Special cases:

- Steady flow $\frac{\partial}{\partial t}\int_{CV} \underline{V}\rho dV = 0$
- Uniform flow across $\underline{A}$:

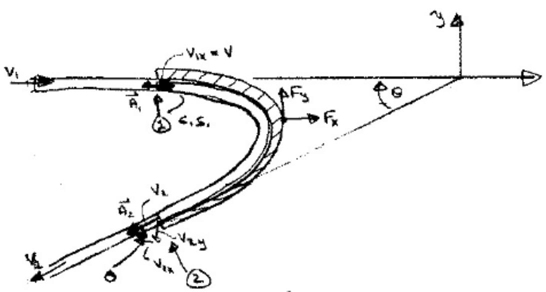
$$\int_{CS} \underline{V}\rho\underline{V}\cdot d\underline{A} = \sum \underline{V}\,\underbrace{\rho\underline{V}\cdot\underline{A}}_{\rho Q=\dot{m}} = \sum\left(\dot{m}\underline{V}\right)_{out} - \sum\left(\dot{m}\underline{V}\right)_{in}$$

- If the flow is also steady,

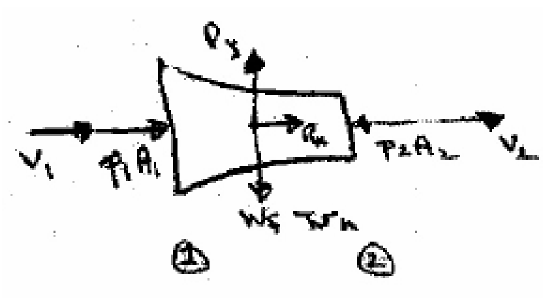
$$\sum F_x = \sum(\dot{m}u)_{out} - \sum(\dot{m}u)_{in}$$

$$\sum F_y = \sum(\dot{m}v)_{out} - \sum(\dot{m}v)_{in}$$

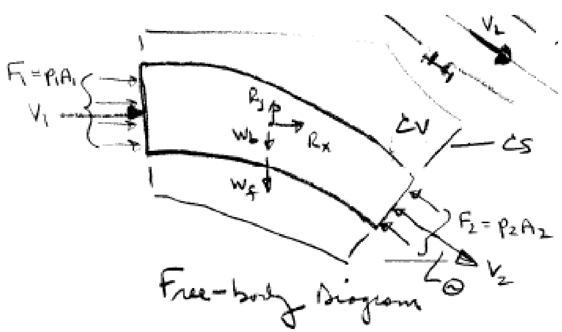
Typical Example (1): Vane

Flow type	$\sum \underline{\boldsymbol{F}}$	$\sum \dot{\boldsymbol{m}} \underline{\boldsymbol{V}}$	Continuity Eq. or Bernoulli Eq.
	$\sum F_x = F_x$ $\sum F_y = F_y - W_{\text{Fluiud}}$ $-W_{\text{Nozzle}}$	x-component: $\underbrace{\dot{m} \cdot (-V_2 \cos\theta)}_{out}$ $-\underbrace{\dot{m} \cdot (V_1)}_{in}$ y-component: $\underbrace{\dot{m} \cdot (-V_2 \sin\theta)}_{out}$ $-\underbrace{\dot{m} \cdot (0)}_{in}$	$p_1 + \frac{\rho V_1^2}{2} + z_1$ $= p_2 + \frac{\rho V_2^2}{2} + z_2$ $p_1 = p_2 = 0$ $z_1 \approx z_2$ $\therefore V_1 = V_2 = A_j$ $V_1 A_1 = V_2 A_2$ $\therefore A_1 = A_2 = A_j$ $\Rightarrow \dot{m} = \rho V_j A_j$

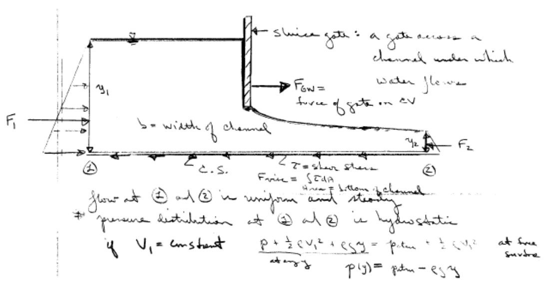
Typical Example (2): Nozzle

Flow type	$\sum \underline{\boldsymbol{F}}$	$\sum \dot{\boldsymbol{m}} \underline{\boldsymbol{V}}$	Continuity Eq. or Bernoulli Eq.
	$\sum F_x = p_1 A_1 + R_x$ $\sum F_y = R_y - W_{Fluiud} - W_{Nozzle}$	x-component: $\underbrace{\dot{m} \cdot (V_2)}_{out} - \underbrace{\dot{m} \cdot (V_1)}_{in}$ y-component: 0	$A_1 V_1 = A_2 V_2$ $p_1 + \frac{\rho V_1^2}{2} = \frac{\rho V_2^2}{2}$ $(\because z_1 = z_1, p_2 = 0)$

Typical Example (3): Bend

Flow type	$\sum \underline{\boldsymbol{F}}$	$\sum \dot{\boldsymbol{m}}\underline{\boldsymbol{V}}$	Continuity Eq. or Bernoulli Eq.
	$\sum F_x = R_x + p_1A_1 - p_2A_2 \cos\theta$ $\sum F_y = R_y + p_2A_2 \sin\theta - W_{\text{Fluiud}} - W_{\text{Nozzle}}$	x-component: $\underbrace{\dot{m}\cdot(V_2 \cos\theta)}_{out} - \underbrace{\dot{m}\cdot(V_1)}_{in}$ y-component: $\underbrace{\dot{m}\cdot(-V_2 \sin\theta)}_{out} - \underbrace{\dot{m}\cdot(0)}_{in}$	$A_1V_1 = A_2V_2$ $p_1 + \frac{\rho V_1^2}{2} + z_1 = p_2 + \frac{\rho V_2^2}{2} + z_2$

Typical Example (4): Sluice Gate

Flow type	$\sum \underline{\boldsymbol{F}}$	$\sum \dot{m}\underline{\boldsymbol{V}}$	Continuity Eq. or Bernoulli Eq.
	$\sum F_x = F_{GW}$ $+\gamma\left(\frac{y_1}{2}\right)(y_1 b)$ $-\gamma\left(\frac{y_2}{2}\right)(y_2 b)$ $\sum F_y = 0$	x-component: $\underbrace{\dot{m}\cdot(V_2)}_{out} - \underbrace{\dot{m}\cdot(V_1)}_{in}$ y-component: $\underbrace{\dot{m}\cdot 0}_{out} - \underbrace{\dot{m}\cdot 0}_{in} = 0$	$V_1(y_1 b) = V_2(y_2 b)$ $\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + h_L$ $(\because p_1 = p_2 = 0;$ $h_L \approx 0)$

Energy Equation

For $B = E$ thus $\beta = e$,

$$\dot{Q} - \dot{W} = \frac{DE_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \underline{V} \cdot d\underline{A}$$

where,

$$e = \breve{u} + e_k + e_p = \breve{u} + \frac{V^2}{2} + gz$$

and

$$\dot{W} = \dot{W}_{\mathrm{s}} + \left(\dot{W}_{\mathrm{fp}} + \dot{W}_{\mathrm{fs}}\right) = \dot{W}_s + \dot{W}_{\mathrm{fp}} = \left(\dot{W}_{\mathrm{t}} - \dot{W}_{\mathrm{p}}\right) + \dot{W}_{\mathrm{fp}}$$

Simplified Energy Equation

For steady, one-dimensional flow:

$$\frac{p_{in}}{\gamma} + \alpha_{in}\frac{V_{in}^2}{2g} + z_{in} + h_p = \frac{p_{out}}{\gamma} + \alpha_{out}\frac{V_{out}^2}{2g} + z_{out} + h_t + h_L$$

- Pump head $h_p = \dot{W}_p/\dot{m}g = \dot{W}_p/\rho Q g = \dot{W}_p/\gamma Q$
- Turbine head $h_t = \dot{W}_t/\dot{m}g = \dot{W}_t/\rho Q g = \dot{W}_t/\gamma Q$
- Head loss $h_L = \mathrm{loss}/g = (\hat{u}_2 - \hat{u}_1)/g - \dot{Q}/\dot{m}g > 0$
- α : kinetic energy correction factor ($\alpha = 1$ for uniform flow across CS)
- V in energy equation refers to average velocity $\bar{V}$

Hydraulic and Energy Grade Lines

Energy Grade Line $EGL = \frac{p}{\gamma} + z + \alpha \frac{V^2}{2g}$

Hydraulic Grade Line $HGL = \frac{p}{\gamma} + z$

$$EGL_{in} + h_p = EGL_{out} + h_t + h_L$$

$EGL_1 = EGL_2 + h_L$
for $h_p = h_t = 0$

EGL = HGL if V = 0

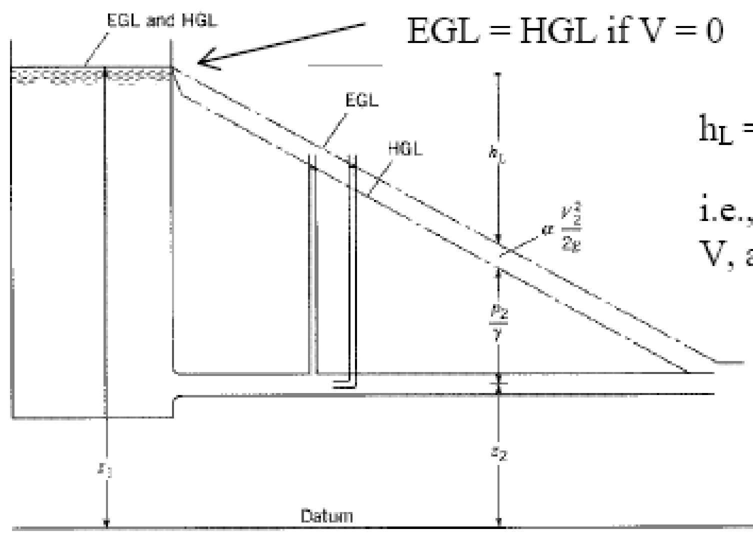


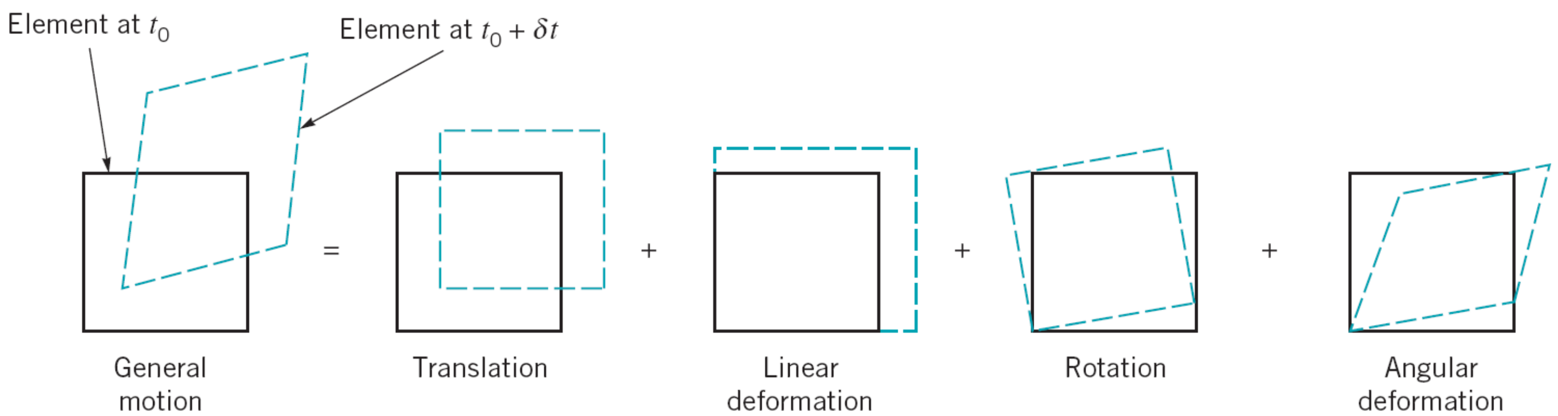
FIGURE 7.4
EGL and HGL in a straight pipe.

$$h_L = f \frac{L}{D} \frac{V^2}{2g}$$

i.e., linear variation in L for D, V, and f constant

f = friction factor
f = f(Re)

Fluid Element Kinematics



- Linear deformation(dilatation): $\nabla \cdot \underline{V}$

  $\Rightarrow$ if the fluid is **incompressible** $\qquad \boldsymbol{\nabla} \cdot \underline{\boldsymbol{V}} = \mathbf{0}$

- Rotation(vorticity): $\underline{\xi} = 2\underline{\omega} = \nabla \times \underline{V}$

  $\Rightarrow$ if the fluid is **irrotational** $\qquad \boldsymbol{\nabla} \times \underline{\boldsymbol{V}} = \mathbf{0}$

- Angular deformation is related to shearing stress

  (e.g., $\tau_{ij} = 2\mu\varepsilon_{ij}$ for Newtonian fluids)

Mass Conservation

For a fluid particle,

$$\lim_{CV\to 0}\left[\int_{CV}\frac{\partial\rho}{\partial t}d\forall+\int_{CS}\rho\underline{V}\cdot d\underline{A}\right]$$

$$=\lim_{CV\to 0}\int_{CV}\left[\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)\right]d\forall=0$$

$$\therefore\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\underline{V}\right)=0$$

For an incompressible flow: $\nabla\cdot\underline{V}=0$

Momentum Conservation

$$\lim_{CV\to 0}\left[\int_{CV}\frac{\partial \underline{V}}{\partial t}\rho d\forall + \int_{CS}\underline{V}\rho\underline{V}\cdot\underline{dA}\right] = \sum \underline{F}$$

or

$$\lim_{CV\to 0}\int_{CV}\rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right)d\forall = \sum \underline{F}$$

$$\therefore \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \sum \underline{f} \qquad (\underline{f} = \underline{F} \text{ per unit volume})$$

$$\Rightarrow \rho\left(\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V}\right) = \underbrace{-\rho g\hat{k}}_{\text{body force due to gravity force}} \underbrace{\underbrace{-\nabla p}_{\text{pressure force}} + \underbrace{\nabla\cdot\tau_{ij}}_{\text{viscous shear force}}}_{\text{surface force}}$$

Navier-Stokes Equations

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right] = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

$$\rho\left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right] = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Cylindrical Coordinates

Continuity:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0$$

Momentum:

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial \theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

Exact Solutions of NS Eqns.

The flow of interest is assumed additionally (than incompressible & Newtonian) as:

1) Steady (i.e., $\boldsymbol{\partial/\partial t = 0}$ for any variable)

2) Parallel such that the y-component of velocity is zero (i.e., $\boldsymbol{v = 0}$)

3) Purely two dimensional (i.e., $\boldsymbol{w = 0}$ and $\boldsymbol{\partial/\partial z = 0}$ for any velocity component)

4) Fully developed (i.e., $\boldsymbol{\partial u/\partial x}$ from the continuity equation and the assumptions 2 and 3)

Boundary Conditions

1) **No-slip condition** ($\underline{V}_{fluid} = \underline{V}_{wall}$; for a stationary wall $\underline{V}_{wall} = 0$)
2) Interface boundary condition ($\underline{V}_A = \underline{V}_B$ and $\tau_{s,A} = \tau_{s,B}$)
3) Free-surface boundary condition ($p_{liquid} = p_{gas}$ and $\tau_{s,fluid} = 0$)

4) Inlet/outlet boundary condition
5) Symmetry boundary condition
6) Initial condition (for unsteady flow problem)

Solving the NS Eqns

1) Set up the problem and geometry (e.g., sketches), identifying all relevant dimensions and parameters.

2) List all appropriate assumptions, approximations, simplifications, and boundary conditions.

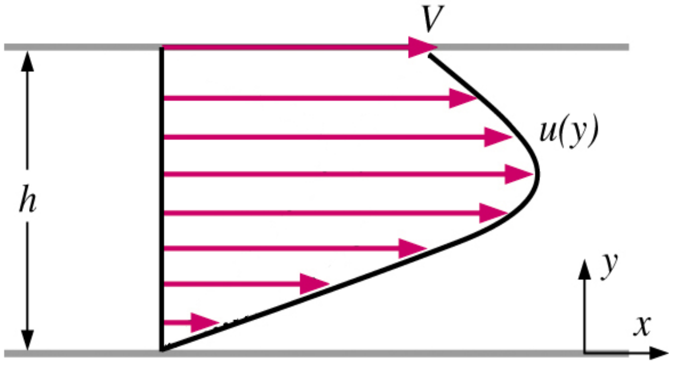
3) Simplify the differential equations of motion (continuity and Navier-Stokes) as much as possible.

4) Integrate the equations, leading to one or more constants of integration

5) Apply boundary conditions to solve for the constants of integration.

6) Verify your results.

Example: Couette Flow



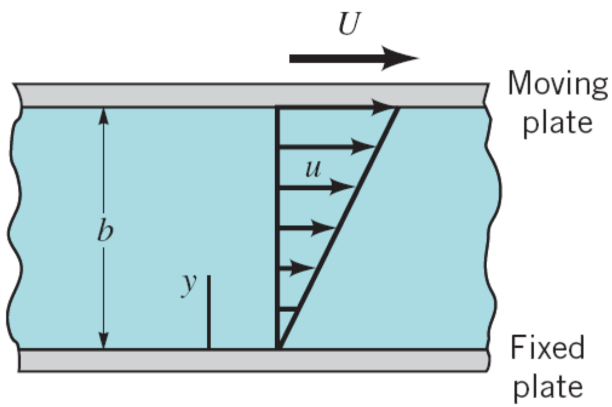
Momentum:

$$0 = \frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2}$$

B.C.: $u(b) = U, u(0) = 0$

$$\therefore u(y) = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) + \frac{yU}{b}$$

Special Case(1): Without $\partial p/\partial x$



Momentum:

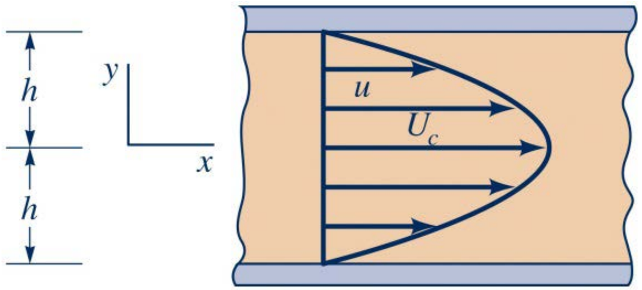
$$0 = \mu \frac{d^2u}{dy^2}$$

B.C.: $u(b) = U, u(0) = 0$

$$\therefore u(y) = \frac{yU}{b}$$

$$\tau_w = \mu \frac{du}{dy}\bigg|_{y=0} = \frac{\mu U}{b}$$

Special Case(2): Both plates fixed



Momentum:

$$0 = \frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2}$$

B.C.: $u(h) = u(-h) = 0$

$$\therefore u(y) = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h)$$

$$q = \int_{-h}^{h} u dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right); \quad \bar{V} = \frac{q}{2h} = -\frac{h^2}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

Buckingham Pi Theorem

- For any physically meaningful equation involving k variables, such as

$$u_1 = f(u_2, u_3, \cdots, u_k)$$

- with minimum number of reference dimensions r, the equation can be rearranged into product of $k - r$ pi terms.

$$\Pi_1 = \phi(\Pi_2, \Pi_3, \cdots, \Pi_{k-r})$$

Repeating Variable Method

Step 1: List all variables that are involved in the problem

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Step 2: Express each of the variables in terms of basic dimensions (either MLT or FLT system)

$$\Delta p_{ell} \doteq FL^{-3}; D \doteq L; \rho \doteq FL^{-4}T^2; \mu \doteq FL^{-2}T; V \doteq LT^{-1}$$

Step 3: Determine the required number of pi terms

$$k - r = 5 - 3 = 2$$

Repeating Variable Method – Contd.

Step 4: Select a number (r) of repeating variables

$$D, V, \rho$$

Step 5: Form a pi term for one of the non-repeating variables

$$\Pi_1 = \Delta p_\ell D^a V^b \rho^c$$

Step 6: Repeat step 5 for each of the remaining non-repeating variables

$$\Pi_2 = \mu D^a V^b \rho^c$$

Repeating Variable Method – Contd.

Step 7: Check all the resulting pi terms to make sure they are dimensionless and independent

$$\Pi_1 = \frac{\Delta p_\ell D}{\rho V^2} \doteq F^0 L^0 T^0; \quad \Pi_2 = \frac{\mu}{DV\rho} \doteq F^0 L^0 T^0$$

Step 8: Express the final form as a relationship among the pi terms

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

Similarity and Model Testing

If all relevant dimensionless parameters have the same corresponding values for model and prototype, flow conditions for a model test are completely similar to those for prototype.

For,

$$\Pi_1 = \phi(\Pi_2, \dots, \Pi_n)$$

if

$$\Pi_{2,model} = \Pi_{2,prototype}$$
$$\vdots$$
$$\Pi_{n,model} = \Pi_{n,prototype}$$

then,

$$\Pi_{1,\mathrm{model}} = \Pi_{1,\mathrm{prototype}}$$

Example: Model Testing

$$\frac{\Delta p_\ell D}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$

- Model design condition (similarity requirements)

$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho V D}{\mu} \quad \Rightarrow \quad V_m = \left(\frac{\rho}{\rho_m}\right)\left(\frac{D}{D_m}\right)\left(\frac{\mu_m}{\mu}\right)V$$

- Prediction equation

$$\frac{\Delta p_\ell D}{\rho V^2} = \frac{\Delta p_{\ell_m} D_m}{\rho_m V_m^2} \quad \Rightarrow \quad \Delta p_\ell = \left(\frac{D_m}{D}\right)\left(\frac{\rho}{\rho_m}\right)\left(\frac{V}{V_m}\right)^2 \Delta p_{\ell_m}$$