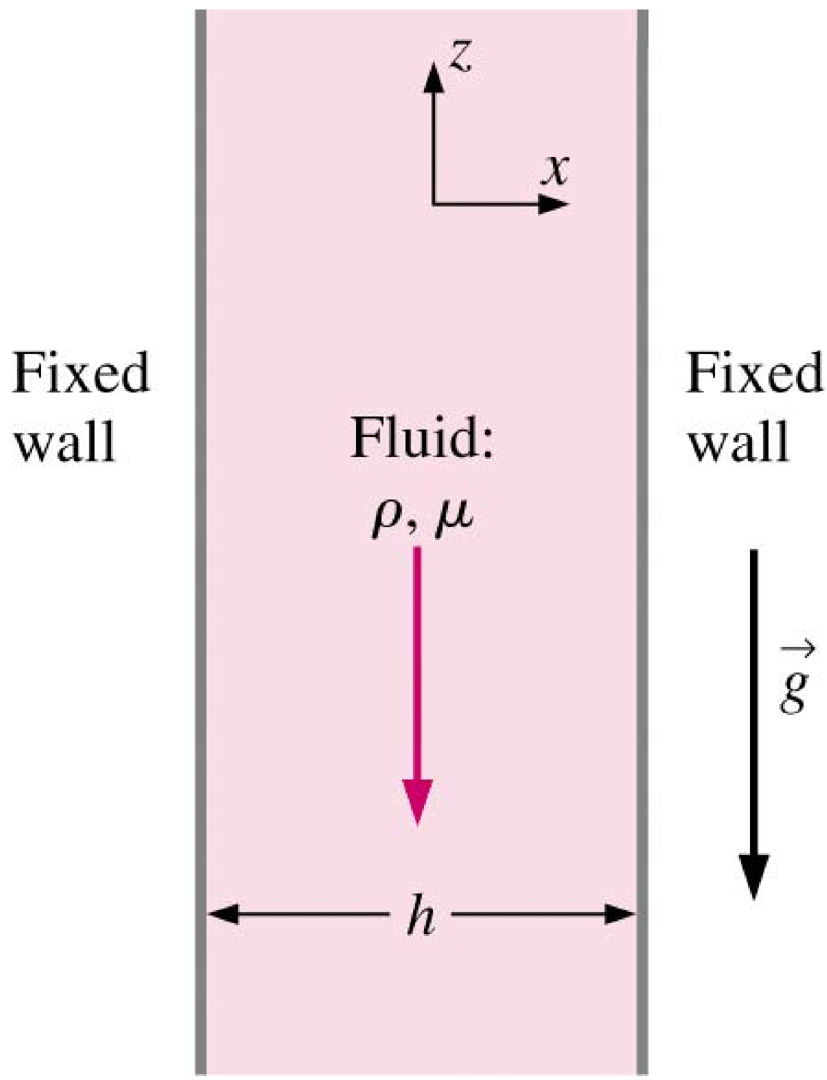
Example 5

Consider steady, incompressible, parallel, laminar flow of a viscous fluid falling between two infinite vertical walls (Fig. 5). The distance between the walls is h, and gravity acts in the negative z-direction (downward in the figure). There is no applied (forced) pressure driving the flow—the fluid falls by gravity alone. The pressure is constant everywhere in the flow field. Calculate the velocity field and sketch the velocity profile using appropriate nondimensionalized variables.



Solution of Example 5:

Make assumptions as;

1. Flow is parallel. → u=0
2. 2D in x-z plane → $\frac{\partial}{\partial y} = 0$
3. Pressure is constant at everywhere. → $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial z} = 0$

Apply these assumptions to Continuity equation and Navier-Stokes equations, then

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \rightarrow \text{use assumption 1} \rightarrow \frac{\partial w}{\partial z} = 0$

NS equations:

x-component: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \rightarrow$ use assumption 1~3 → All terms vanish

z-component: $u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - g$

→ use assumption 1~3 and simplified Continuity equation → $\frac{\partial^2 w}{\partial x^2} = \frac{\rho g}{\mu}$

From simplified z-component of NS equation, we obtain

$$w(x) = \frac{\rho g}{2\mu}x^2 + Ax + B$$

Apply boundary conditions (i.e. no-slip condition) such as;

At x=0 → $w(0)$=0

At x=h → $w(h)$=0

in order to get the coefficients A and B. → $A = -\frac{\rho g h}{2\mu}, B = 0$

Finally, $w(x) = \frac{\rho g}{2\mu}\left(x - \frac{h}{2}\right)^2 - \frac{\rho g h^2}{8\mu}$

To non-dimensionalize this velocity profile, set non-dmensional variables as;

$$x^* = \frac{x}{h} \rightarrow x = hx^*$$

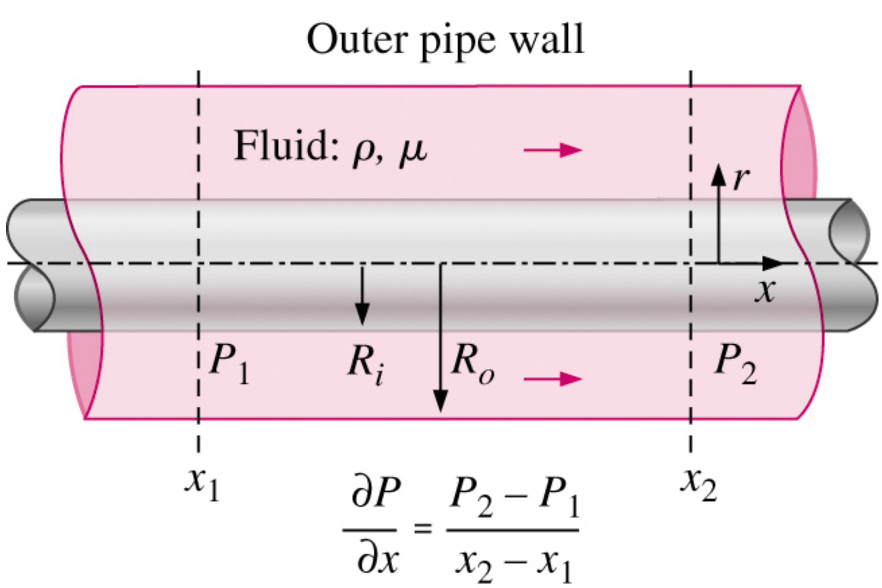
$$w^* = \frac{w\mu}{\rho g h^2} \rightarrow w = \frac{\rho g h^2}{\mu} w^*$$

Plug these variables into the velocity profile, the obtain

$$w^*(x^*) = \frac{1}{2}\left(x^{*2} - x^*\right)$$

Example 6

Consider steady, incompressible, laminar flow of a Newtonian fluid in an infinitely long round pipe annulus of inner radius Ri and outer radius Ro (Fig. P6). Ignore the effects of gravity. A constant negative pressure gradient _P/_x is applied in the x-direction, $(\partial P/\partial x)=(P_2-P_1)/(x_2-x_1)$, where x1 and x2 are two arbitrary locations along the x-axis, and P1 and P2 are the pressures at those two locations. The pressure gradient may be caused by a pump and/or gravity. Note that we adopt a modified cylindrical coordinate system here with x instead of z for the axial component, namely, (r,θ,x) and (u_r,u_θ,u). Derive an expression for the velocity field in the annular space in the pipe.



Solution of Example 6:

Make assumptions as;

1. Flow is axisymmetric $\rightarrow \dfrac{\partial}{\partial \theta} = 0, u_\theta = 0$
2. Flow is parallel $\rightarrow u_r = 0$

Apply these assumptions to Continuity equation and Navier-Stokes equations in cylindrical coordinates, then

Continuity: $\dfrac{1}{r}\dfrac{\partial (r u_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{\partial u}{\partial x} = 0 \rightarrow \text{use assumption 1 and 2} \rightarrow \dfrac{\partial u}{\partial x} = 0$

NS equations:

• r - component:

$$\rho\left(u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) - \frac{u_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u}{\partial x^2}\right]$$

$\rightarrow$ use assumption1,2 and simplified Continutity equation $\rightarrow -\dfrac{\partial p}{\partial r} = 0$

• θ - component:

$$\rho\left(u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r u_\theta}{r} + u\frac{\partial u_\theta}{\partial x}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right) - \frac{u_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial x^2}\right]$$

$\rightarrow$ use assumption1,2 and simplified Continutity equation $\rightarrow$ All terms vanish

• x - component:

$$\rho\left(u_r \frac{\partial u}{\partial x} + \frac{u_\theta}{r}\frac{\partial u}{\partial \theta} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right]$$

$\rightarrow$ use assumption1,2 and simplified Continuity equation $\rightarrow -\dfrac{\partial p}{\partial x} + \mu\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) = 0$

Make use of simplified x-component of NS equation;

$\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right) = \dfrac{r}{\mu}\dfrac{\partial p}{\partial x} \rightarrow$ Integrate both sides by $r \rightarrow r\dfrac{\partial u}{\partial r} = \dfrac{r^2}{2\mu}\dfrac{\partial p}{\partial x} + C_1 \rightarrow$ Integrate both sides by r again

$\rightarrow u(r) = \dfrac{r^2}{4\mu}\dfrac{\partial p}{\partial x} + C_1 \ln r + C_2$

Apply boundary conditions such as;

At $r = R_i$: $u = 0$

At $r = R_o$: $u = 0$

Then, obtain the constants C_1 and C_2 as;

$$C_1 = -\frac{\left(R_o^2 - R_i^2\right)}{4\mu \ln\frac{R_o}{R_i}}\frac{dP}{dx} \qquad C_2 = \frac{\left(R_o^2 \ln R_i - R_i^2 \ln R_o\right)}{4\mu \ln\frac{R_o}{R_i}}\frac{dP}{dx}$$

Therefore, the final solution is

$$u = \frac{1}{4\mu}\frac{dP}{dx}\left(r^2 + \frac{R_i^2 \ln\frac{r}{R_o} - R_o^2 \ln\frac{r}{R_i}}{\ln\frac{R_o}{R_i}}\right)$$