Creeping Flow

Basic assumption for creeping flow: the inertia terms are negligible in the momentum equation if Re<<1.

If we nondimensionalize the NS equation with the variables

$$x^* = \frac{x}{L}, \quad \mathbf{V}^* = \frac{\mathbf{V}}{U}, \quad t^* = \frac{tU}{L}, \quad p^* = \frac{p - p_\infty}{\mu U / L}$$

(Note pressure difference scales with $\mu U/L$ instead of ρU^2 due to the basic assumption of creeping flow)
we obtain the flowing dimensionless momentum equation:

$$\mathrm{Re}\frac{D\mathbf{V}^*}{Dt^*} = -\nabla^* p^* + \nabla^{*2}\mathbf{V}^*$$

Since Re<<1, we have:

$$\nabla^* p^* \approx \nabla^{*2}\mathbf{V}^*$$

In dimensional form:

$$\nabla p = \mu \nabla^2 \mathbf{V} \qquad (1)$$

It should be combined with incompressible continuity equation:

$$\nabla \cdot \mathbf{V} = 0 \qquad (2)$$

(1) and (2) are the basic equations for creeping flows.

Taking the curl and them gradient of (1), we obtain two additional useful relations, i.e. both the vorticity and the pressure satisfy Laplace's equation in creeping flow:

$$\nabla^2 \boldsymbol{\omega} = 0 \qquad (3)$$

$$\nabla^2 p = 0 \qquad (4)$$

Since $\omega = -\nabla^2 \psi$ in 2D Stokes flow, where ψ is the stream function, (3) may be rewritten in terms of ψ:

$$\nabla^4 \psi = 0 \qquad (5)$$

<u>Applications of Creeping Flow Theory</u>

1. Fully developed duct flow: inertia terms also vanish
2. Flow about immersed bodies: usually small particles
3. Flow in narrow but variable passages: lubrication theory
4. Flow through porous media: groundwater movement

<u>Drag on an Object in Creeping Flow</u>:

Since the inertia (density) is truly negligible, we have:

$$F = \text{total drag force} = f\left(U, \mu, L\right) \qquad (6)$$

Follow the step-by-step method discussed in Chapter 7:

There are 4 parameters in the problem $(n = 4)$

There are 3 primary dimensions: M, L, T, $(j = 4)$

We expect only one Pi since $k = n - j = 4 - 3 = 1$, and the Pi must equal to a constant. The final result is

$$F = const \cdot \mu UL \qquad (7)$$

<u>Drag on a Sphere in Creeping Flow</u>:

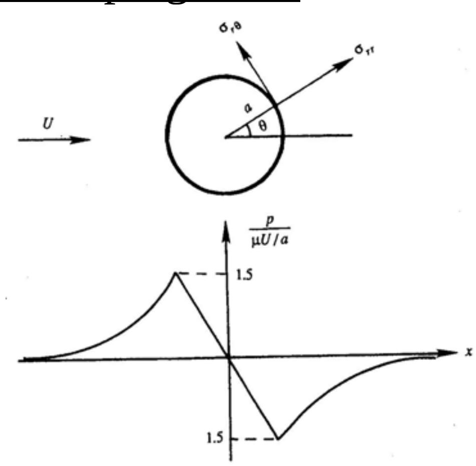

Fig. 1 Creeping flow over a sphere: the viscous stress components at the surface and the pressure distribution in an axial plane

Consider creeping motion of a stream of speed U about a solid sphere of radius a. It is convenient to use spherical polar coordinates (r,θ) with $\theta=0$ in the direction of U. The only component of vorticity in this axisymmetric problem is ω_φ, and is given by

$$\omega_\varphi=\frac{1}{r}\left[\frac{\partial(ru_\theta)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right] \tag{8}$$

In axisymmetric flows we can define the Stokes stream function ψ, and it is defined in spherical polar coordinates as

$$u_r\equiv\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},\quad u_\theta\equiv-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} \tag{9}$$

In terms of stream function, the vorticity becomes

$$\omega_\varphi=-\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \tag{10}$$

Plugging (10) in the governing equation in terms of vorticity

$$\nabla^2\omega_\varphi=0 \tag{11}$$

we get

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi=0 \tag{12}$$

The boundary conditions on the above equation are:

$$\psi(a,\theta)=0 \qquad \text{(i.e. } u_r=0 \text{ at surface)} \tag{13}$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{(a,\theta)}=0 \qquad \text{(i.e. } u_\theta=0 \text{ at surface)} \tag{14}$$

$$\psi(\infty,\theta)=\frac{1}{2}Ur^2\sin^2\theta \qquad \text{(i.e. uniform flow at } \infty\text{)} \tag{15}$$

The last condition follows from the fact that the stream function for a uniform flow is $\frac{1}{2}Ur^2\sin^2\theta$ in spherical polar coordinates.

The upstream condition (15) suggests a separable solution of the form

$$\psi = f(r)\sin^2\theta \tag{16}$$

Substitution of this into the (12) gives

$$f^{(4)} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0 \tag{17}$$

whose solution is

$$f = Ar^4 + Br^2 + Cr + \frac{D}{r} \tag{18}$$

The upstream BC (15) requires that $A = 0$ and $B = \frac{U}{2}$

The surface BC (13) and (14) gives $C = -\frac{3Ua}{4}$ and $D = \frac{Ua^3}{4}$

The solution then reduces to

$$\psi = Ur^2\sin^2\theta\left(\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right) \tag{19}$$

The velocity components can then be found as

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = U\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) \tag{20}$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -U\sin\theta\left(1 - \frac{3a}{4r} + \frac{a^3}{4r^3}\right) \tag{21}$$

With u_r and u_θ known, the pressure is found by integrating the momentum equation (1):

$$p = p_\infty - \frac{3\mu aU}{2r^2}\cos\theta \tag{22}$$

The shear stress distribution in the fluid is given by

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial}{\partial} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\frac{\mu U\sin\theta}{r}\left(\frac{3a^3}{2r^3}\right) \tag{23}$$

The total drag is found by integrating pressure and shear around the sphere surface:

$$
\begin{aligned}
F &= -\int_0^{\pi} \tau_{r\theta}\Big|_{r=a} \sin\theta dA - \int_0^{\pi} p\Big|_{r=a} \cos\theta dA \\
&= -\int_0^{\pi} \tau_{r\theta}\Big|_{r=a} \sin\theta\left(2\pi a^2 \sin\theta d\theta\right) - \int_0^{\pi} p\Big|_{r=a} \cos\theta\left(2\pi a^2 \sin\theta d\theta\right) \\
&= 4\pi\mu Ua + 2\pi\mu Ua = 6\pi\mu Ua \qquad (24)
\end{aligned}
$$

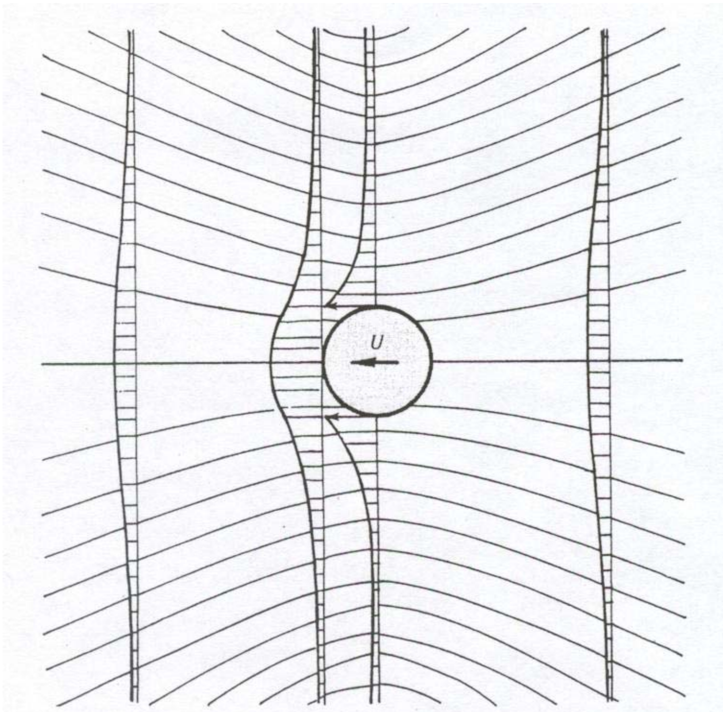

Fig. 2 Streamlines and velocity distributions in Stokes' solution of creeping flow due to a moving sphere. Note the upper stream and downstream symmetry, a result of complete neglect of nonlinearity

The proper drag coefficient should obviously be

$$\frac{F}{\mu Ua} = 6\pi = const \qquad (25)$$

but everyone uses the inertia type of definition

$$C_D = \frac{2F}{\rho U^2 (area)} = \frac{2F}{\rho U^2 \left(\pi a^2\right)} = \frac{24}{\mathrm{Re}} \qquad (26)$$

where

$$\mathrm{Re} = \frac{2a\rho U}{\mu} \tag{27}$$

Oseen's Improvement:

The Stokes solution for a sphere is not valid at large distance from the body, because the advective terms are not negligible compared to the viscous terms at these distances. Oseen provided a cure to Stokes' solution by partly accounting for the inertia terms at large distances. He made the substitutions

$$u = U + u', \quad v = v', \quad w = w' \tag{28}$$

where (u', v', w') are the Cartesian components of the perturbation velocity, small at large distances. Substituting these, the advective term of the x-momentum becomes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial y} = U\frac{\partial u'}{\partial x} + \left(u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial y}\right) \tag{29}$$

Neglecting the quadratic terms, the equation of motion becomes

$$\rho U\frac{\partial u_i'}{\partial x} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u' \tag{30}$$

Where u' represents (u', v', w'). This is called Oseen's equation, and the approximation involved is called Oseen's approximation. In essence, the Oseen approximation linearizes the advective term $\mathbf{u}\cdot\nabla\mathbf{u}$ by $U\cdot\frac{\partial \mathbf{u}}{\partial x}$, whereas the Stokes approximation drops advection altogether. Near the body both approximations have the same order of accuracy. However, the Oseen approximation is better in the far field where the velocity is only slightly different than U. The Oseen equations provide a lowest order solution that is uniformly valid everywhere in the flow field.

The boundary conditions for a moving sphere are

$$u' = v' = w' = 0 \qquad \text{at infinity} \tag{31}$$

$$u' = -U, \quad v' = w' = 0 \qquad \text{at surface} \tag{32}$$

The solution found by Oseen is

$$\frac{\psi}{Ua^2} = \left(\frac{r^2}{2a^2} + \frac{a}{4r}\right)\sin^2\theta - \frac{3}{\mathrm{Re}}(1+\cos\theta)\left\{1 - \exp\left[-\frac{\mathrm{Re}}{4}\frac{r}{a}(1-\cos\theta)\right]\right\} \tag{33}$$

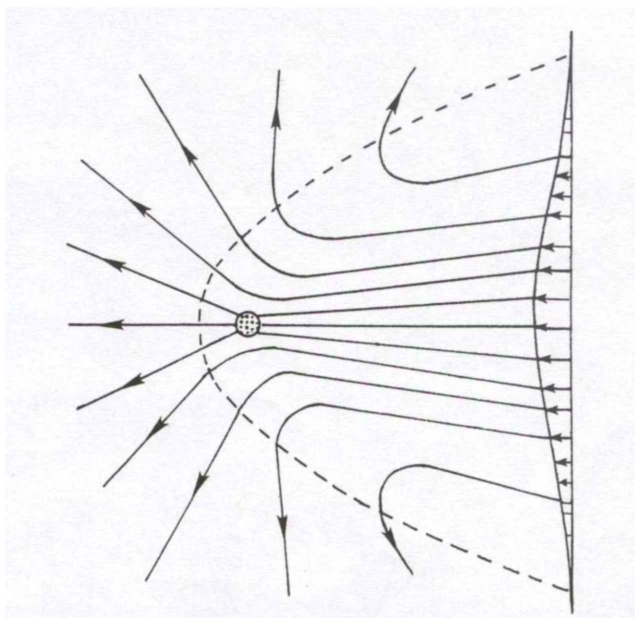

Fig. 3 Streamlines and velocity distributions in Oseen’s solution of creeping flow due to a moving sphere. Note the upper stream and downstream asymmetry, a result of partial accounting for advection in the far field

The Oseen approximation predicts that the drag coefficient is

$$C_D = \frac{24}{\mathrm{Re}}\left(1 + \frac{3}{16}\mathrm{Re}\right) \tag{34}$$

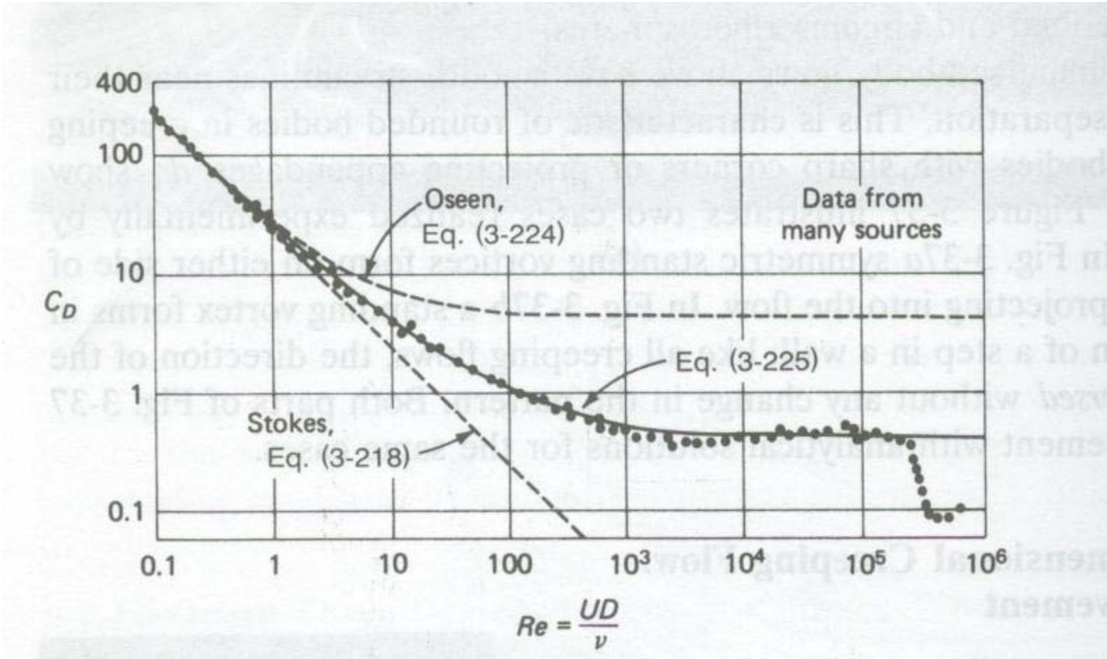

Fig. 4 Comparison of experiment, theory, and empirical formulas for drag coefficients of a sphere; empirical formulas can be found in White's Viscous Flow

<u>Example: Terminal Velocity of a Particle from a Volcano</u>

A volcano has erupted, spewing stones, steam, and ash several thousand feet into the atmosphere. After some time, the particles begin to settle to the ground. Consider a nearly spherical ash particle of diameter 50 mm, falling in air whose temperature is -50°C and whose pressure is 55 kPa. The density of the particle is 1240 kg/m3. Estimate the terminal velocity of this particle at this altitude.

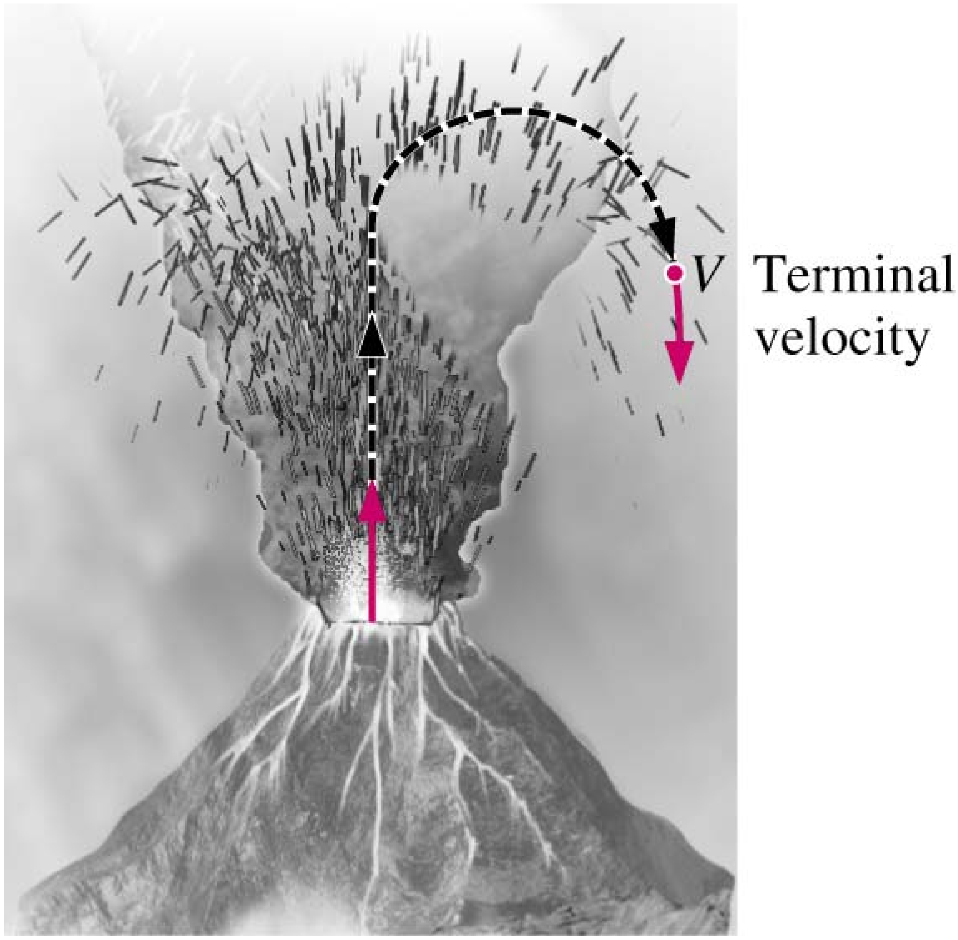

Fig. 5 Small ash particles spewed from a volcano eruption settle slowly to the ground; the creeping flow approximation is reasonable for this type of flow field

Assumptions:

1 The Reynolds number is very small (we will need to verify this assumption after we obtain the solution).

2 The particle is spherical.

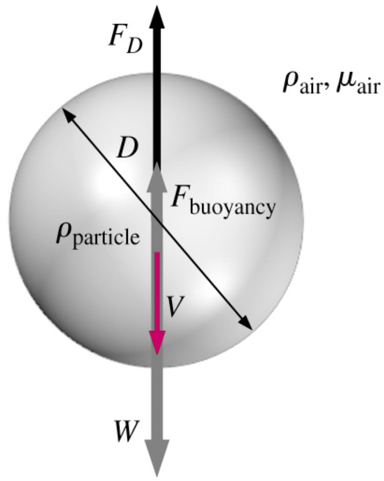

Fig.6 A particle falling at a steady terminal velocity has no acceleration; therefore, its weight is balanced by aerodynamic drag and the buoyancy force acting on the particle

Once the falling particle has reached its terminal velocity, the net downward force (weight) balances the net upward force (aerodynamic drag + buoyancy), as illustrated in the above figure.

Downward force

$$F_{down} = W = \pi \frac{D^3}{6} \rho_{particle} g \tag{35}$$

Upward force

$$F_{up} = F_D + F_{buoyancy} = 3\pi\mu VD + \pi \frac{D^3}{6} \rho_{air} g \tag{36}$$

At given temperature and pressure, the air density can be calculated using the ideal gas law:

$$\rho_{air} = \frac{P}{RT} = 0.8588\, kg/m^3$$

Since viscosity is a very weak function of pressure, we use the value at -50°C and atmospheric pressure, $\mu = 1.474 \times 10^{-5}\, kg/m \cdot s$

We equate (35) and (36) and solve for terminal velocity:

$$V = \frac{D^2}{18\mu}\left(\rho_{particle} - \rho_{air}\right) g = 0.115\, m/s \tag{36}$$

Finally we verify that the Reynolds number is small enough that creeping flow is an appropriate approximation:

$$\mathrm{Re} = \frac{\rho_{air} V D}{\mu} = 0.335 < 1$$

Thus the Reynolds number is less than 1, but certainly not much less than 1.