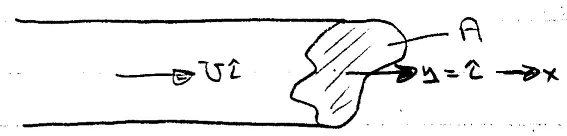
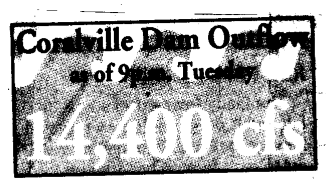
Chapter 5 Mass, Bernoulli, and Energy Equations

5.1 Flow Rate and Conservation of Mass

1. cross-sectional area oriented normal to velocity vector (simple case where $V \perp A$)





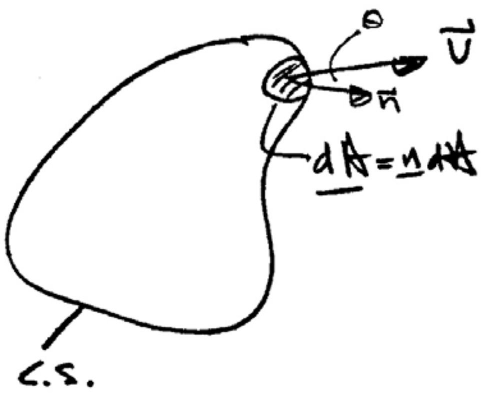
Fall 93

U = constant: Q = volume flux = UA [m/s × m^2 = m^3/s]

U ≠ constant: $Q = \int_A U dA$

Similarly the mass flux = $\dot{m} = \int_A \rho U dA$

2. general case



$$Q = \int_{CS} \underline{V} \cdot \underline{n} dA$$

$$= \int_{CS} |\underline{V}| \cos\theta dA$$

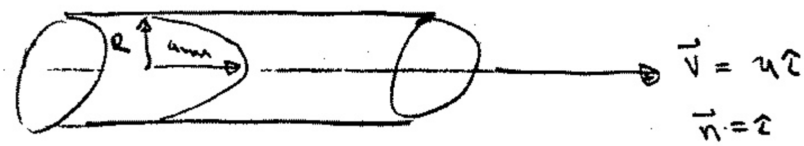
$$\dot{m} = \int_{CS} \rho(\underline{V} \cdot \underline{n}) dA$$

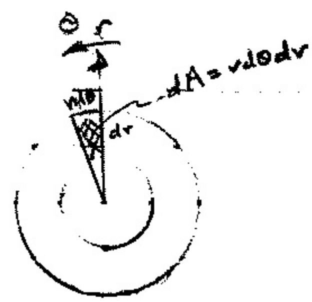
average velocity: $\overline{V} = \dfrac{Q}{A}$

Example:
At low velocities the flow through a long circular tube, i.e. pipe, has a parabolic velocity distribution (actually paraboloid of revolution).

$$u = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

i.e., centerline velocity





a) find Q and $\overline{V}$

$$Q = \int_A \underline{V}\cdot\underline{n}\,dA = \int_A u\,dA$$

$$\int_A u\,dA = \int_0^{2\pi}\int_0^R u(r)\,r\,d\theta\,dr$$

$$= 2\pi\int_0^R u(r)\,r\,dr$$

$dA = 2\pi r dr$

$u = u(r)$ and not θ $\therefore$ $\int_0^{2\pi} d\theta = 2\pi$

$$Q = 2\pi\int_0^R u_{max}\left(1-\left(\frac{r}{R}\right)^2\right)rdr \quad = \frac{1}{2}u_{max}\pi R^2$$

$$\overline{V} = \frac{1}{2}u_{max}$$

<u>Continuity Equation</u>

RTT can be used to obtain an integral relationship expressing conservation of mass by defining the extensive property B = M such that β = 1.

B = M = mass
β = dB/dM = 1

General Form of Continuity Equation

$$\frac{dM}{dt} = 0 = \frac{d}{dt}\int_{CV}\rho d\cancel{V} + \int_{CS}\rho \underline{V}\cdot\underline{dA}$$

or

$$\underbrace{\int_{CS}\rho\underline{V}\cdot\underline{dA}}_{} \quad = \quad \underbrace{-\frac{d}{dt}\int_{CV}\rho d\cancel{V}}_{}$$

net rate of outflow of mass across CS — rate of decrease of mass within CV

Simplifications:

1. Steady flow: $-\frac{d}{dt}\int_{CV}\rho d\forall = 0$

2. $\underline{V}$ = constant over discrete $\underline{dA}$ (flow sections):

$$\int_{CS}\rho\underline{V}\cdot\underline{dA} = \sum_{CS}\rho\underline{V}\cdot\underline{A}$$

3. Incompressible fluid (ρ = constant)

$$\int_{CS}\underline{V}\cdot\underline{dA} = -\frac{d}{dt}\int_{CV}d\forall \qquad \text{conservation of volume}$$

4. Steady One-Dimensional Flow in a Conduit:

$$\sum_{CS}\rho\underline{V}\cdot\underline{A} = 0$$

$$-\rho_1V_1A_1 + \rho_2V_2A_2 = 0$$

for ρ = constant $\quad Q_1 = Q_2$

Some useful definitions:

Mass flux $\dot{m} = \int_A \rho\underline{V}\cdot\underline{dA}$

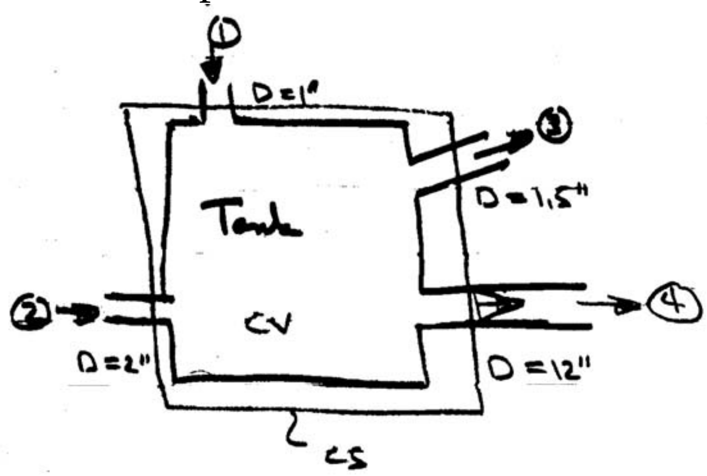
Volume flux $Q = \int_A \underline{V}\cdot\underline{dA}$

Average Velocity $\overline{V} = Q/A$

Average Density $\bar{\rho} = \frac{1}{A}\int \rho dA$

Note: $\dot{m} \neq \bar{\rho}Q$ unless ρ = constant

Example



*Steady flow

*$V_{1,2,3}$ = 50 fps

*@ ¯ V varies linearly
from zero at wall to
V_{max} at pipe center

*find $\dot{m}_4$, Q_4, V_{max}

*water, ρ_w = 1.94 slug/ft^3

$$\int_{CS} \rho \underline{V} \cdot \underline{dA} = 0 = -\cancelto{0}{\frac{d}{dt}\int_{CV} \rho d\forall}$$

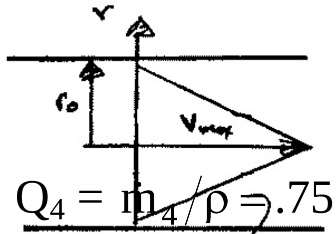
i.e., $-\rho_1 V_1 A_1 - \rho_2 V_2 A_2 + \rho_3 V_3 A_3 + \overbrace{\boxed{\rho \int_{A_4} V_4 dA_4}}^{\dot{m}_4} = 0$

ρ = const. = 1.94 lb-s^2 /ft^4 = 1.94 slug/ft^3

$\dot{m}_4 = \rho\int V_4 dA_4 = \rho V(A_1 + A_2 - A_3)$ $\quad V_1=V_2=V_3=V=50$f/s

$$= \frac{1.94}{144}\times 50\times\frac{\pi}{4}\left(1^2 + 2^2 - 1.5^2\right)$$

= 1.45 slugs/s



$Q_4 = \dot{m}_4/\rho = .75 \text{ ft}^3/\text{s}$

$$= \int_{A_4} V_4 dA_4 \quad V = v_{max}(1 - r/r_o), \quad dA_4 = r dr d\theta$$

velocity profile

$$Q_4 = \int_0^{r_o}\int_0^{2\pi} \underbrace{V_{max}\left(1 - \frac{r}{r_o}\right)}_{V_4 \neq V_4(\theta)} \underbrace{r d\theta dr}_{dA_4}$$

$$= 2\pi \int_0^{r_o} V_{max}\left(1 - \frac{r}{r_o}\right) r dr$$

$$\overline{V}_4 = \frac{Q}{A} = \frac{\frac{1}{3}\pi r_o^2 V_{max}}{\pi r_o^2} = \frac{1}{3}V_{max}$$

$$= 2\pi V_{max} \int_0^{r_o}\left[r - \frac{r^2}{r_o}\right] dr$$

$$= 2\pi V_{max}\left[\left.\frac{r^2}{2}\right|_0^{r_0} - \left.\frac{r^3}{3r_o}\right|_0^{r_o}\right]$$

$$= 2\pi V_{max} r_o^{\,2}\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}\pi r_o^{\,2} V_{max}$$

$$V_{max} = \frac{Q_4}{\frac{1}{3}\pi r_o^{\,2}} = 2.86 \text{ fps}$$

5.2 Mechanical Energy, Efficiency, Bernoulli Equations, Application, and Limitations

Assume irrotational, inviscid, and incompressible flow = ideal flow theory

Also, assume steady flow

$\underline{\Omega} = \nabla \times \underline{V} = 0 \;\Rightarrow\; \underline{V} = \nabla\varphi$ irrotational

$\underline{a} = -\nabla(p/\rho + gz),\;\; \nabla \cdot \underline{V} = 0$ inviscid, incompressible

$$\underline{a} = \underline{V}\cdot\nabla\underline{V} = \nabla\tfrac{1}{2}\underline{V}\cdot\underline{V} + \cancelto{0}{\underline{V}\times(\nabla\times\underline{V})} \qquad \text{steady}$$

$$= \nabla\tfrac{1}{2}V^2 \qquad V^2 = \underline{V}\cdot\underline{V}$$

$$\nabla\frac{1}{2}V^2 = -\nabla\left(\frac{p}{\rho} + gz\right)$$

$$\nabla\left(\frac{1}{2}V^2 + \frac{p}{\rho} + gz\right) = 0$$

i.e., $p + \tfrac{1}{2}\rho V^2 + \gamma z = B = \text{constant}$

$$p_1 + \tfrac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \tfrac{1}{2}\rho V_2^2 + \gamma z_2$$

Also, from continuity and irrotational

$\nabla \cdot \underline{V} = 0$ $\qquad\underline{V} = \nabla\varphi = \dfrac{\partial\varphi}{\partial x}\hat{i} + \dfrac{\partial\varphi}{\partial y}\hat{j} + \dfrac{\partial\varphi}{\partial z}\hat{k}$

$\nabla \cdot \nabla\varphi = 0$ $\qquad\varphi$ = velocity potential

$\nabla^2 \varphi = 0$ i.e., governing differential equation for φ is Laplace equation

Application of Bernoulli's Equation

Stagnation Tube

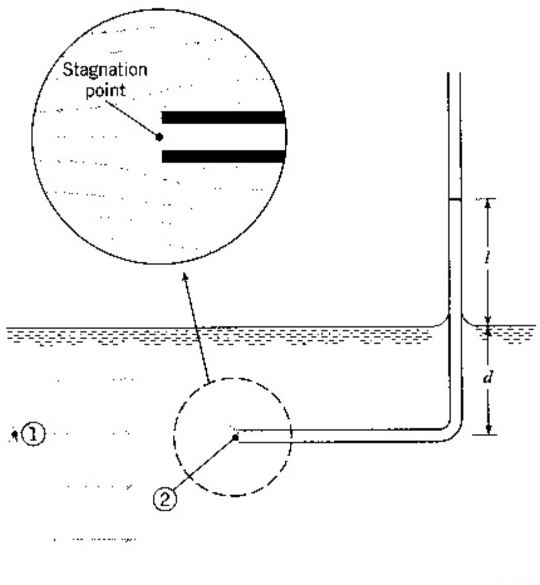


FIGURE 5.5
Stagnation tube.

at ⌐ V = 0

$$p_1 + \rho \frac{V_1^2}{2} = p_2 + \rho \frac{V_2^2}{2}$$

$z_1 = z_2$

$$V_1^2 = \frac{2}{\rho}(p_2 - p_1)$$

$p_1 = \gamma d$
$p_2 = \gamma(\ell + d)$

$V_2 = 0$
gage

$$= \frac{2}{\rho}(\gamma \ell)$$

$$V_1 = \sqrt{2g\ell}$$

Limited by length of tube and need for free surface reference

Pitot Tube

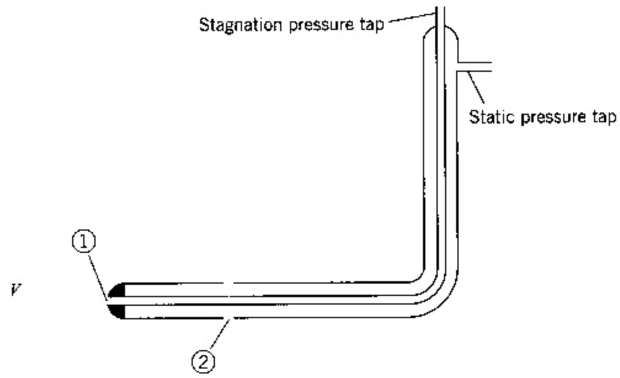


FIGURE 5.6
Pitot tube.

$$\underbrace{\frac{p_1}{\gamma}+\cancelto{0}{\frac{V_1^2}{2g}}+z_1}_{\text{at 1}}=\underbrace{\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2}_{\text{at 2}}$$

$$V_2=\left\{2g\left[\underbrace{\left(\frac{p_1}{\gamma}+z_1\right)}_{h_1}-\underbrace{\left(\frac{p_2}{\gamma}+z_2\right)}_{h_2}\right]\right\}^{1/2} \qquad V_1=0$$

$$V=V_2=\sqrt{2g(h_1-h_2)}$$

h = piezometric head
$h_1 - h_2$ from manometer
or pressure gage

for gas flow $\frac{\Delta p}{\gamma} >> \Delta z$

$$V=\sqrt{\frac{2\Delta p}{\rho}}$$

5.3 Derivation of the Energy Equation

The First Law of Thermodynamics

The difference between the heat added to a system and the work done by a system depends only on the initial and final states of the system; that is, depends only on the change in energy E: principle of conservation of energy

$$\Delta E = Q - W$$

ΔE = change in energy
Q = heat added to the system
W = work done by the system

$E = E_u + E_k + E_p$ = total energy of the system

potential energy
kinetic energy
Internal energy due to molecular motion

The differential form of the first law of thermodynamics expresses the rate of change of E with respect to time

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

rate of work being done by system
rate of heat transfer to system

Energy Equation for Fluid Flow

The energy equation for fluid flow is derived from Reynolds transport theorem with

B_{system} = E = total energy of the system (extensive property)

β = E/mass = e = energy per unit mass (intensive property)
= u + e_k + e_p

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV} \rho e d\cancel{V} + \int_{CS} \rho e \underline{V}\cdot\underline{dA}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho(u + e_k + e_p) d\cancel{V} + \int_{CS} \rho(u + e_k + e_p)\underline{V}\cdot\underline{dA}$$

This can be put in a more useable form by noting the following:

$$e_k = \frac{\text{Total KE of mass with velocity V}}{\text{mass}} = \frac{\Delta M V^2/2}{\Delta M} = \frac{V^2}{2} \qquad V^2 = |\underline{V}|$$

$$e_p = \frac{E_p}{\Delta M} = \frac{\gamma \Delta\cancel{V} z}{\rho\Delta\cancel{V}} = gz \qquad \text{(for } E_p \text{ due to gravity only)}$$

$$\dot{Q} - \dot{W} = \frac{d}{dt}\int_{CV} \rho\left(\frac{V^2}{2} + gz + u\right) d\cancel{V} + \int_{Cs} \rho\left(\frac{V^2}{2} + gz + u\right)\underline{V}\cdot\underline{dA}$$

rate of work done by system

rate of change of energy in CV

flux of energy out of CV (ie, across CS)

rate of heat transfer to sysem

<u>Rate of Work Components</u>: $\dot{W} = \dot{W}_s + \dot{W}_f$

For convenience of analysis, work is divided into shaft work W_s and flow work W_f

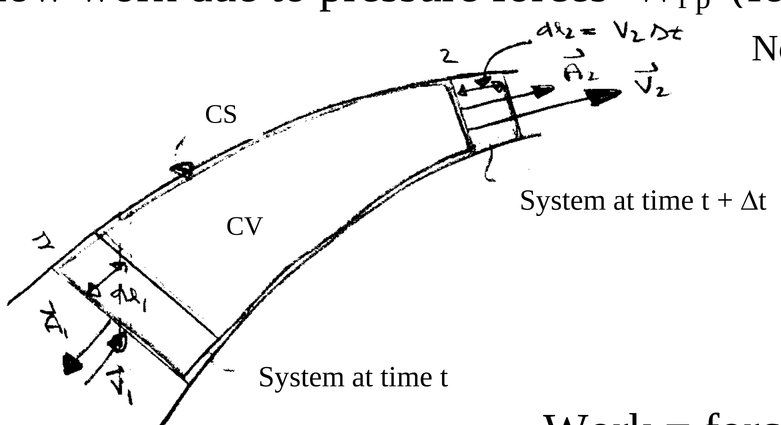
W_f = net work done on the surroundings as a result of normal and tangential stresses acting at the control surfaces

$= W_{f\ pressure} + W_{f\ shear}$

W_s = any other work transferred to the surroundings usually in the form of a shaft which either takes energy out of the system (turbine) or puts energy into the system (pump)

Flow work due to pressure forces $W_{f\,p}$ (for system)

Note: here $\underline{V}$ uniform over $\underline{A}$



Work = force × distance

at 2 $W_2 = p_2A_2 \times V_2\Delta t$ (on surroundings)

rate of work $\Rightarrow$ $\dot{W}_2 = p_2A_2V_2 = p_2\underline{V}_2 \cdot \underline{A}_2$

at 1 $W_1 = -p_1A_1 \times V_1\Delta t$

$\dot{W}_1 = p_1\underline{V}_1 \cdot \underline{A}_1$

neg. sign since pressure force on surrounding fluid acts in a direction opposite to the motion of the system boundary

In general,

$$\dot{W}_{fp} = p\underline{V}\cdot\underline{A}$$

for more than one control surface and $\underline{V}$ not necessarily uniform over $\underline{A}$:

$$\dot{W}_{fp} = \int_{CS} p\underline{V}\cdot\underline{dA} = \int_{CS}\rho\left(\frac{p}{\rho}\right)\underline{V}\cdot\underline{dA}$$

$$\dot{W}_{f} = \dot{W}_{fp} + \dot{W}_{fshear}$$

Basic form of energy equation

$$\dot{Q} - \dot{W}_s - \dot{W}_{fshear} - \int_{CS}\rho\left(\frac{p}{\rho}\right)\underline{V}\cdot\underline{dA}$$

$$= \frac{d}{dt}\int_{CV}\rho\left(\frac{V^2}{2} + gz + u\right)d\forall + \int_{CS}\rho\left(\frac{V^2}{2} + gz + u\right)\underline{V}\cdot\underline{dA}$$

$$\dot{Q} - \dot{W}_s - \cancel{\dot{W}_{fshear}} = \frac{d}{dt}\int_{CV}\rho\left(\frac{V^2}{2} + gz + u\right)d\forall$$

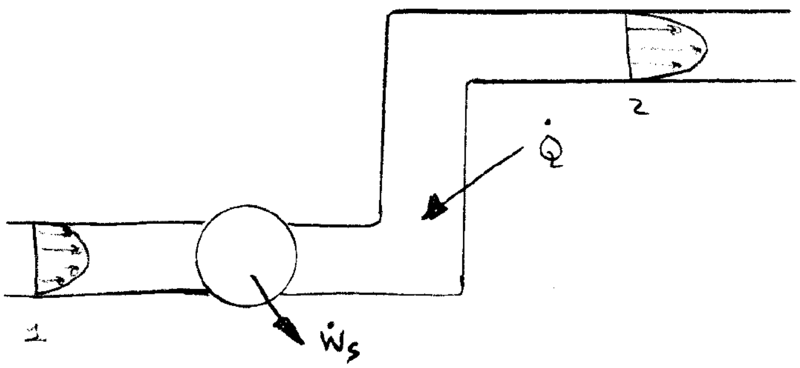
$$+ \int_{CS}\rho\left(\frac{V^2}{2} + gz + \underbrace{u + \frac{p}{\rho}}_{h=\text{enthalpy}}\right)\underline{V}\cdot\underline{dA}$$

Usually this term can be eliminated by proper choice of CV, i.e. CS normal to flow lines. Also, at fixed boundaries the velocity is zero (no slip condition) and no shear stress flow work is done. Not included or discussed in text!

5.4 Simplified Forms of the Energy Equation

Energy Equation for Steady One-Dimensional Pipe Flow

Consider flow through the pipe system as shown



Energy Equation (steady flow)

$$\dot{Q} - \dot{W}_s = \int_{CS} \rho \left(\frac{V^2}{2} + gz + \frac{p}{\rho} + u \right) \underline{V} \cdot \underline{dA}$$

$$\dot{Q} - \dot{W}_s + \int_{A_1} \left(\frac{p_1}{\rho} + gz_1 + u_1 \right) \rho_1 V_1 A_1 + \int_{A_1} \frac{\rho_1 V_1^3}{2} dA_1$$

$$= \int_{A_2} \left(\frac{p_2}{\rho} + gz_2 + u_2 \right) \rho_2 V_2 A_2 + \int_{A_2} \frac{\rho_2 V_2^3}{2} dA_2$$

*Although the velocity varies across the flow sections the streamlines are assumed to be straight and parallel; consequently, there is no acceleration normal to the streamlines and the pressure is hydrostatically distributed, i.e., $p/\rho + gz =$ constant.

*Furthermore, the internal energy u can be considered as constant across the flow sections, i.e. T = constant. These quantities can then be taken outside the integral sign to yield

$$\dot{Q} - \dot{W}_s + \left(\frac{p_1}{\rho} + gz_1 + u_1\right)\rho\int_{A_1} V_1 dA_1 + \rho\int_{A_1} \frac{V_1^3}{2} dA_1$$

$$= \left(\frac{p_2}{\rho} + gz_2 + u_2\right)\rho\int_{A_2} V_2 dA_2 + \rho\int_{A_2} \frac{V_2^3}{2} dA_2$$

Recall that $\quad Q = \int \underline{V} \cdot \underline{dA} = \overline{V} A$

So that $\quad \rho\int \underline{V} \cdot \underline{dA} = \rho\overline{V} A = \dot{m} \qquad$ mass flow rate

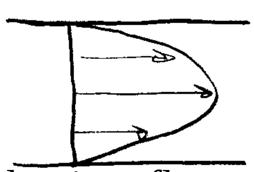
Define: $$\underbrace{\rho\int_A \frac{V^3}{2} dA}_{\text{K.E. flux}} = \alpha \underbrace{\frac{\rho \overline{V}^3 A}{2}}_{\text{K.E. flux for } V=\overline{V}=\text{constant across pipe}} = \alpha \frac{\overline{V}^2}{2} \dot{m}$$

i.e., $$\alpha = \frac{1}{A}\int_A \left(\frac{V}{\overline{V}}\right)^3 dA \text{ = kinetic energy correction factor}$$

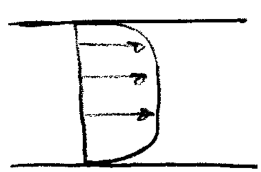
$$\dot{Q} - \dot{W} + \left(\frac{p_1}{\rho} + gz_1 + u_1 + \alpha_1 \frac{\overline{V}_1^2}{2}\right)\dot{m} = \left(\frac{p_2}{\rho} + gz_2 + u_2 + \alpha_2 \frac{\overline{V}_2^2}{2}\right)\dot{m}$$

$$\frac{1}{\dot{m}}\left(\dot{Q} - \dot{W}\right) + \frac{p_1}{\rho} + gz_1 + u_1 + \alpha_1 \frac{\overline{V}_1^2}{2} = \frac{p_2}{\rho} + gz_2 + u_2 + \alpha_2 \frac{\overline{V}_2^2}{2}$$

note that: $\quad \alpha = 1$ if V is constant across the flow section
$\qquad \alpha > 1$ if V is nonuniform



laminar flow $\alpha = 2$



turbulent flow $\alpha = 1.05 \sim 1$ may be used

<u>Shaft Work</u>

Shaft work is usually the result of a turbine or a pump in the flow system. When a fluid passes through a turbine, the fluid is doing shaft work on the surroundings; on the other hand, a pump does work on the fluid

$\dot{W}_s = \dot{W}_t - \dot{W}_p$ where $\dot{W}_t$ and $\dot{W}_p$ are magnitudes of power $\left(\dfrac{\text{work}}{\text{time}}\right)$

Using this result in the energy equation and deviding by g results in

$$\underbrace{\frac{\dot{W}_p}{\dot{m}g} + \frac{p_1}{\gamma} + z_1 + \alpha_1 \frac{\overline{V}_1^2}{2} = \frac{\dot{W}_t}{\dot{m}g} + \frac{p_2}{\gamma} + z_2 + \alpha_2 \frac{\overline{V}_2^2}{2}}_{\text{mechanical part}} + \underbrace{\frac{u_2 - u_1}{g} - \frac{\dot{Q}}{\dot{m}g}}_{\text{thermal part}}$$

Note: each term has dimensions of length

Define the following:

$$h_p = \frac{\dot{W}_p}{\dot{m}g} = \frac{\dot{W}_p}{\rho Q g} = \frac{\dot{W}_p}{\gamma Q}$$

$$h_t = \frac{\dot{W}_t}{\dot{m}g}$$

$$h_L = \frac{u_2 - u_1}{g} - \frac{\dot{Q}}{\dot{m}g} = \text{head loss}$$

Head Loss

In a general fluid system a certain amount of mechanical energy is converted to thermal energy due to viscous action. This effect results in an increase in the fluid internal energy. Also, some heat will be generated through energy dissipation and be lost (i.e. $-\dot{Q}$). Therefore the term

$$h_L = \frac{u_2 - u_1}{g} - \underbrace{\frac{\dot{Q}}{g\dot{m}}}_{\text{from } 2^{nd} \text{ law}} > 0$$

represents a loss in mechanical energy due to viscous stresses

Note that adding $\dot{Q}$ to system will not make $h_L = 0$ since this also increases Δu. It can be shown from 2^{nd} law of thermodynamics that $h_L > 0$.

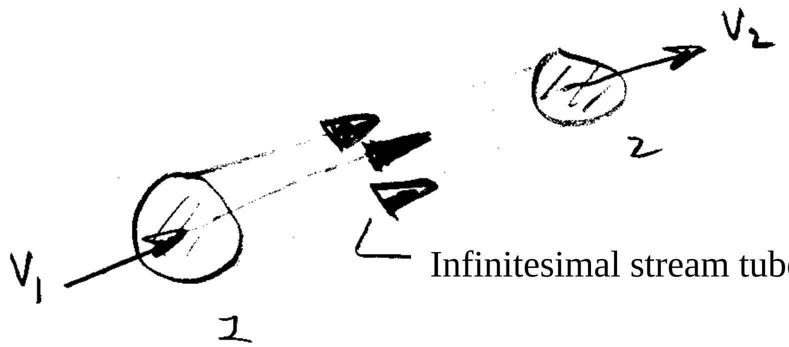
Drop — over $\overline{V}$ and understand that V in energy equation refers to average velocity.

Using the above definitions in the energy equation results in (steady 1-D incompressible flow)

$$\underbrace{\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_t + h_L}_{\text{form of energy equation used for this course!}}$$

Comparison of Energy Equation and Bernoulli Equation:

Apply energy equation to a stream tube without any shaft work



Infinitesimal stream tube $\Rightarrow \alpha_1=\alpha_2=1$

Energy eq : $$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+h_L$$

•If $h_L = 0$ (i.e., $\mu = 0$) we get Bernoulli equation and conservation of mechanical energy along a streamline

•Therefore, energy equation for steady 1-D pipe flow can be interpreted as a modified Bernoulli equation to include viscous effects (h_L) and shaft work (h_p or h_t)

5.5 Concept of Hydraulic and Energy Grade Lines

$$\frac{p_1}{\gamma}+\alpha_1\frac{V_1^2}{2g}+z_1+h_p=\frac{p_2}{\gamma}+\alpha_2\frac{V_2^2}{2g}+z_2+h_t+h_L$$

Define $\left.\begin{aligned} \text{HGL} &= \frac{p}{\gamma}+z \\ \text{EGL} &= \frac{p}{\gamma}+z+\alpha\frac{V^2}{2g}\end{aligned}\right\}$ point-by-point application is graphically displayed

HGL corresponds to pressure tap measurement + z
EGL corresponds to stagnation tube measurement + z

$EGL_1 = EGL_2 + h_L$
for $h_p = h_t = 0$

EGL = HGL if V = 0

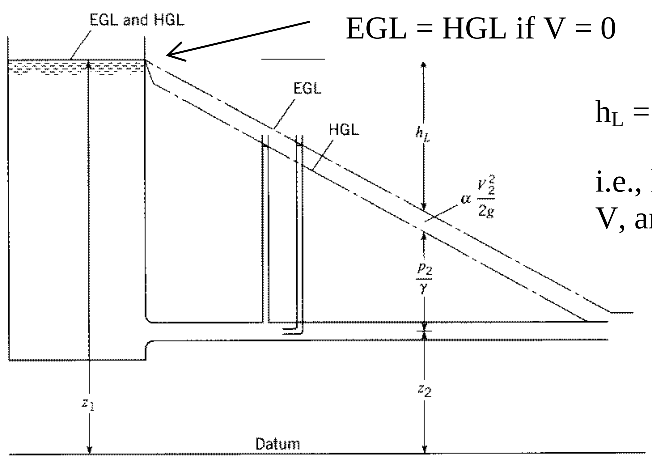


FIGURE 7.4
EGL and HGL in a straight pipe.

$h_L = f\frac{L}{D}\frac{V^2}{2g}$
i.e., linear variation in L for D, V, and f constant

f = friction factor
f = f(Re)

pressure tap: $\frac{p_2}{\gamma}=h$

stagnation tube: $\frac{p_2}{\gamma}+\alpha\frac{V_2^2}{2g}=h$

h = height of fluid in tap/tube

$EGL_1 + h_p = EGL_2 + h_t + h_L$

$EGL_2 = EGL_1 + \underbrace{h_p - h_t}_{\text{abrupt change due to } h_p \text{ or } h_t} - h_L$

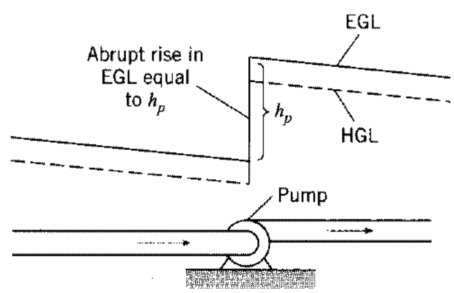
$f\frac{L}{D}\frac{V^2}{2g}$

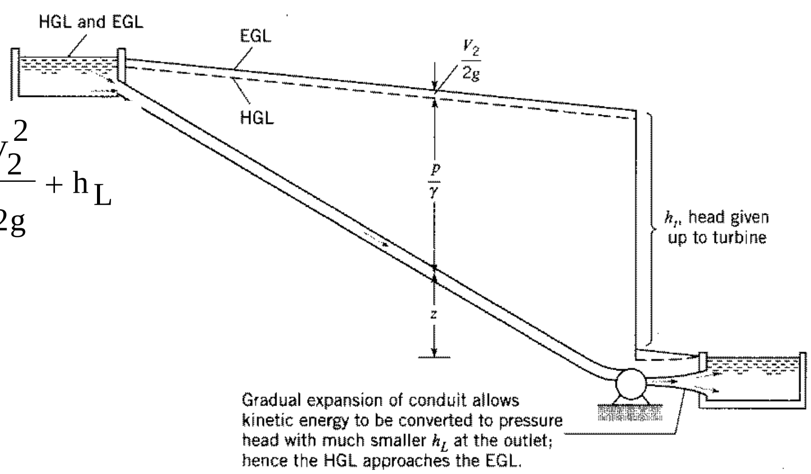
Helpful hints for drawing HGL and EGL

1. EGL = HGL + $\alpha V^2/2g$ = HGL for V = 0

2.&3. $h_L = f\dfrac{L}{D}\dfrac{V^2}{2g}$ in pipe means EGL and HGL will slope downward, except for abrupt changes due to h_t or h_p

FIGURE 7.5
Rise in EGL and HGL due to pump.





$$\frac{p_1}{\gamma} + z_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} + h_L$$

$HGL_2 = EGL_1 - h_L$

$h_L = \dfrac{V^2}{2g}$ for abrupt expansion

Drop in EGL and HGL due to turbine.

4. p = 0 $\Rightarrow$ HGL = z

5. for $h_L = f\frac{L}{D}\frac{V^2}{2g}$ = constant $\times$ L

i.e., linearly increased for increasing L with slope $f \; V^2$

EGL/HGL slope downward

6. for change in D $\Rightarrow$ change in V

i.e.
$$\left.\begin{aligned} V_1A_1 &= V_2A_2 \\ V_1\frac{\pi D_1^2}{4} &= V_2\frac{\pi D_2^2}{4} \\ V_1D_1^2 &= V_1D_2^2 \end{aligned}\right\} \Rightarrow$$
change in distance between HGL & EGL and slope change due to change in h_L

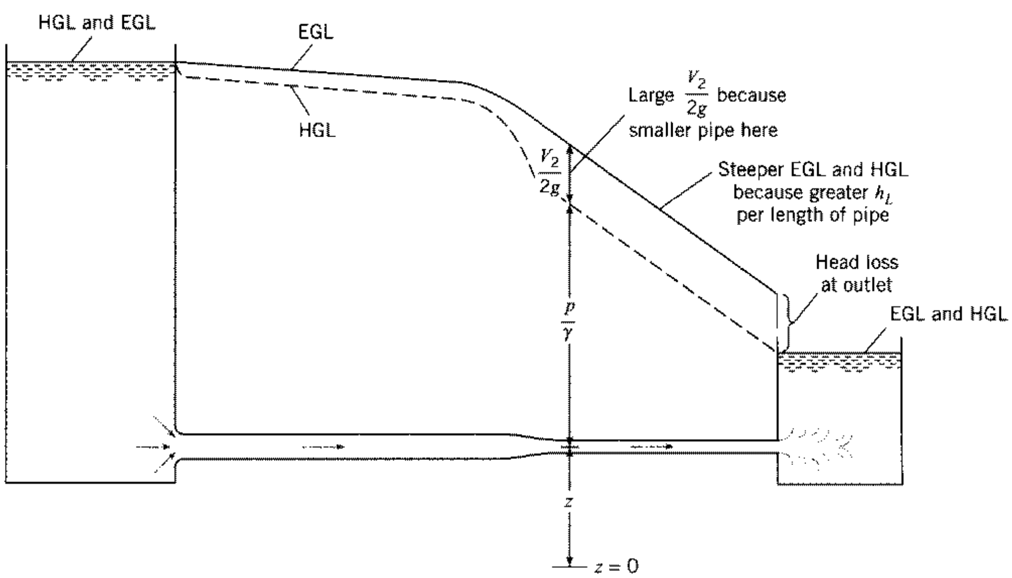


FIGURE 7.8
Change in EGL and HGL due to change in diameter of pipe.

7. If HGL < z then $p/\gamma < 0$ i.e., cavitation possible

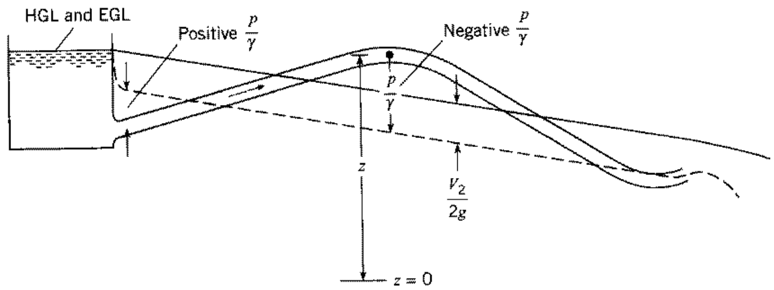


FIGURE 7.9
Subatmospheric pressure when pipe is above HGL.

condition for cavitation:

$$p = p_{va} = 2000\frac{N}{m^2}$$

gage pressure $p_{va,g} = p_A - p_{atm} \approx -p_{atm} = -100{,}000\frac{N}{m^2}$

$$\frac{p_{va,g}}{\gamma} \approx -10m$$

9810 N/m^3

Summary of the Energy Equation

The energy equation is derived from Reynolds Transport Theorem with

B = E = total energy of the system

β = e = E/M = energy per unit mass

$$= \underbrace{u}_{\text{internal}} + \underbrace{\frac{1}{2}V^2}_{\text{KE}} + \underbrace{gz}_{\text{PE}}$$

$$\frac{dE}{dt} = \frac{d}{dt}\int_{CV}\rho e d\forall + \int_{CS}\rho e \underline{V}\cdot\underline{dA} = \underbrace{\dot{Q}}_{\text{heat add}} - \underbrace{\dot{W}}_{\text{work done}}$$

from 1st Law of Thermodynamics

$$\dot{W} = \underbrace{\dot{W}_s}_{\substack{\text{shaft work}\\ \text{done on or}\\ \text{by system}\\ \text{(pump or}\\ \text{turbine)}}} + \underbrace{\dot{W}_p}_{\substack{\text{pressure}\\ \text{work done}\\ \text{on CS}}} + \underbrace{\overset{\text{Neglected in text presentation}}{\cancel{\dot{W}_v}}}_{\substack{\text{Viscous stress}\\ \text{work on CS}}}$$

$$\dot{W}_p = \int_{CV} p\underline{V}\cdot\underline{dA} = \int_{CS}\rho(p/\rho)\underline{V}\cdot\underline{dA}$$

$$\dot{W}_s = \dot{W}_t - \dot{W}_p$$

$$\dot{Q} - \dot{W}_t + \dot{W}_p = \frac{d}{dt}\int_{CV} \rho e d\forall + \int_{CS} \rho(e + p/e)\underline{V} \cdot \underline{dA}$$

$$e = u + \frac{1}{2}V^2 + gz$$

For steady 1-D pipe flow (one inlet and one outlet):

1) Streamlines are straight and parallel
$\Rightarrow$ $p/\rho + gz$ = constant across CS

2) T = constant $\Rightarrow$ u = constant across CS

3) define $\alpha = \frac{1}{A}\int_{CS}\left(\frac{V}{\overline{V}}\right)^3 dA$ = KE correction factor

$$\Rightarrow \quad \frac{\rho}{2}\int V^3 dA = \alpha\frac{\rho\overline{V}^3}{2}A = \alpha\frac{\overline{V}^2}{2}\dot{m}$$

$$\overbrace{\frac{p_1}{\gamma} + \alpha_1\frac{V_1^2}{2g} + z_1 + h_p = \frac{p_2}{\gamma} + \alpha_2\frac{V_2^2}{2g} + z_2 + h_t}^{\text{mechanical energy}} + \underbrace{h_L}_{\text{Thermal energy}}$$

$$h_p = \dot{W}_p / \dot{m}g$$

$$h_t = \dot{W}_t / \dot{m}g$$

$$h_L = \frac{u_2 - u_1}{g} - \frac{\dot{Q}}{\dot{m}g} = \text{head loss}$$

> 0 represents loss in mechanical energy due to viscosity

<u>Note: each term has</u> units of length

V is average velocity (vector dropped) and corrected by α

4.15 METHOD OF SOLUTION OF FLOW PROBLEMS

For the solutions of problems of liquid flow there are two fundamental equations, the equation of continuity (3.10) and the energy equation in one of the forms from Eqs. (4.5) to (4.10). The following procedure may be employed:

1. Choose a datum plane through any convenient point.
2. Note at what sections the velocity is known or is to be assumed. If at any point the section area is great compared with its value elsewhere, the velocity head is so small that it may be disregarded.
3. Note at what points the pressure is known or is to be assumed. In a body of liquid at rest with a free surface the pressure is known at every point within the body. The pressure in a jet is the same as that of the medium surrounding the jet.
4. Note whether or not there is any point where all three terms, pressure, elevation, and velocity, are known.
5. Note whether or not there is any point where there is only one unknown quantity.

It is generally possible to write an energy equation that will fulfill conditions 4 and 5. If there are two unknowns in the equation, then the continuity equation must be used also. The application of these principles is shown in the following illustrative examples.

Illustrative Example 4.7 A pipeline with a pump leads to a nozzle as shown in the accompanying figure. Find the flow rate when the pump develops a head of 80 ft. Assume that the head loss in the 6-in-diameter pipe may be expressed by $h_L = 5V_6^2/2g$, while the head loss in the 4-in-diameter pipe is $h_L = 12V_4^2/2g$. Sketch the energy line and hydraulic grade line, and find the pressure head at the suction side of the pump.

Select the datum as the elevation of the water surface in the reservoir. Note from continuity that

$$V_6 = (\tfrac{3}{6})^2 V_3 = 0.25V_3 \qquad \text{and} \qquad V_4 = (\tfrac{3}{4})^2 V_3 = 0.563V_3$$

where V_3 is the jet velocity. Writing an energy equation from the surface of the reservoir to the jet,

$$\left(z_1 + \frac{p_1}{\gamma} + \frac{V_1^2}{2g}\right) - h_{L_6} + h_p - h_{L_4} = z_3 + \frac{p_3}{\gamma} + \frac{V_3^2}{2g}$$

$$0 + 0 + 0 - 5\frac{V_6^2}{2g} + 80 - 12\frac{V_4^2}{2g} = 10 + 0 + \frac{V_3^2}{2g}$$

Express all velocities in terms of V_3:

$$-\frac{5(0.25V_3)^2}{2g} + 80 - 12\frac{(0.563V_3)^2}{2g} = 10 + \frac{V_3^2}{2g}$$

$$V_3 = 29.7 \text{ fps}$$

$$Q = A_3V_3 = \frac{\pi}{4}\left(\frac{3}{12}\right)^2 29.7 = 1.45 \text{ cfs}$$

Head loss in suction pipe:

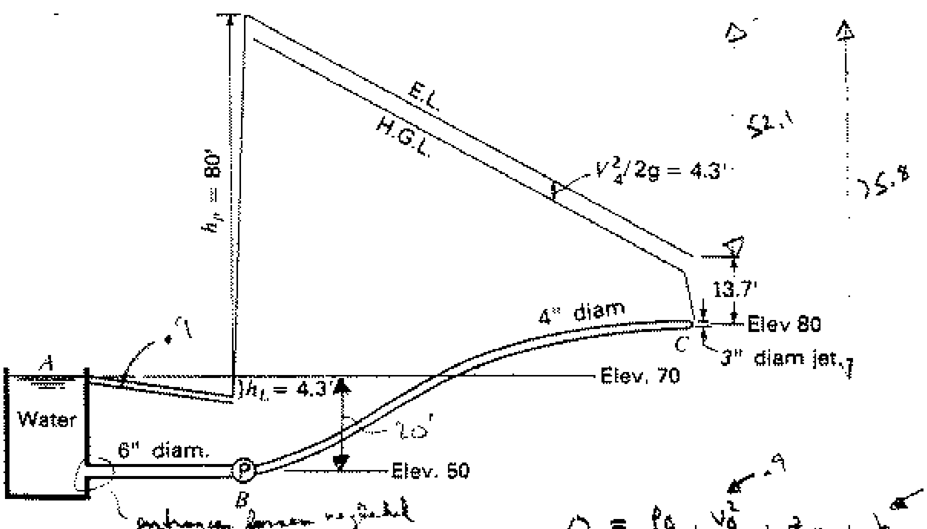
$$h_L = 5\frac{V_6^2}{2g} = \frac{5(0.25V_3)^2}{2g} = \frac{0.312V_3^2}{2g}$$
$$= 4.3 \text{ ft}$$

Head loss in discharge pipe:

$$h_L = 12\frac{V_4^2}{2g} = \frac{12(0.563V_3)^2}{2g} = 52.1 \text{ ft}$$

$$\frac{V_3^2}{2g} = 13.7 \text{ ft} \qquad \frac{V_4^2}{2g} = 4.3 \text{ ft} \qquad \frac{V_6^2}{2g} = 0.86 \text{ ft} \approx 0.9 \text{ ft}$$

The energy line and hydraulic grade line are drawn on the figure to scale. Inspection of the figure shows that the pressure head on the suction side of the pump is $p_B/\gamma = 14.8$ ft. Likewise, the pressure head at any point in the pipe may be found if the figure is to scale.



Illustrative Example 4.7

$$0 = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B + h_L \Rightarrow \frac{p_B}{\gamma} = 14.8$$

z_A .9 4.3 −20

Illustrative Example 4.8 Given the two-dimensional flow as shown in the accompanying figure. Determine the flow rate. Assume no head loss.

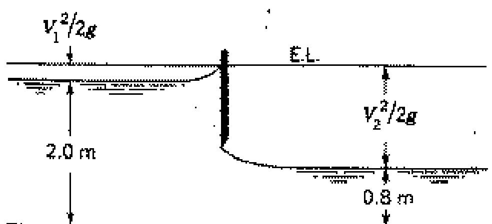
+ over pump:

$$h_p + \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + z_B = \frac{p}{\gamma} + \frac{V^2}{2g} + z + h_L$$

14.8 + .9 − 20

−4.3

75.7



Illustrative Example 4.8