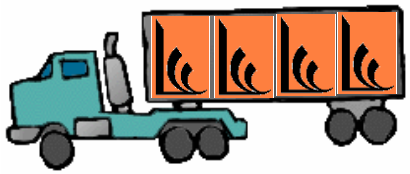
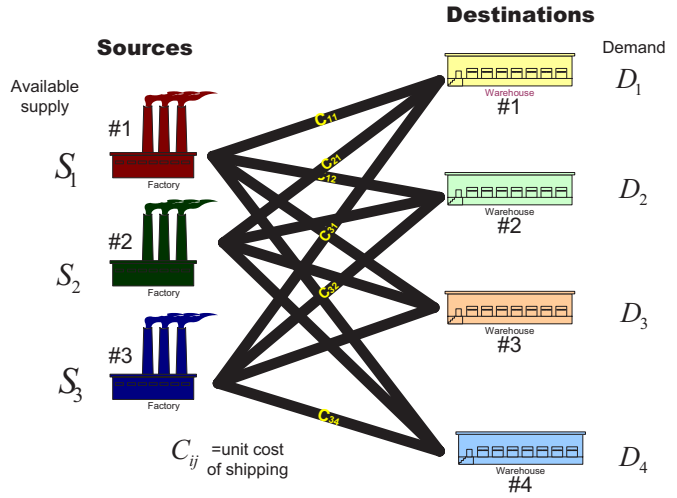
The Transportation Problem



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Let X_{ij} = quantity shipped from source i to destination j

C_{ij} = cost per unit shipped on the route from i to j

S_i = quantity available at source i

D_j = quantity required at destination j

Minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}X_{ij}$

subject to

$$\sum_{j=1}^{n} X_{ij} \le S_i \quad \textit{for every source } i=1,\ldots m$$

$$\sum_{i=1}^{m} X_{ij} \ge D_j \quad \textit{for every destination } j=1,\ldots n$$

$$X_{ij} \ge 0 \quad \textit{for every } i \text{ \& } j$$

The Transportation Problem

This is a ***Linear Programming*** problem with

- $m+n$ *inequality constraints*
- $m \times n$ *nonnegative variables*

***Balanced* Transportation Problem:**

total supply = total demand

This implies that all available supplies be shipped, and no more is received than required, i.e.,

$$\sum_{j=1}^{n} X_{ij} = S_i \quad \forall i \qquad \text{and} \qquad \sum_{i=1}^{m} X_{ij} = D_j \quad \forall j$$

The **LP Simplex Method** assumes equality constraints (after adding slack & surplus variables when appropriate).

The ***rank*** (# of linearly independent equations) of the equality constraints of a balanced transportation problem is m + n – 1 because of a redundancy:

Sum all of the ***supply constraints***:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij} = \sum_{i=1}^{m} S_i$$

Sum all of the ***demand constraints***:

$$\sum_{j=1}^{n}\sum_{i=1}^{m} X_{ij} = \sum_{j=1}^{n} D_j$$

Since $\sum_{i=1}^{m} S_i = \sum_{j=1}^{n} D_j$, these are ***identical constraints***!

In other words…. in a ***balanced*** problem,

if we ignore one of the constraints, e.g. the last demand constraint,

then it will be satisfied by any solution of the remaining system of equations:

All supplies *must* be shipped, and

the other destinations will receive a total of exactly $\sum_{j=1}^{n-1} D_j$,

so that the remaining $\sum_{i=1}^{m} S_i - \sum_{j=1}^{n-1} D_j = D_n$ *must* be shipped to the last destination!

The number of *basic* variables is therefore m + n – 1.

The ***ordinary*** Simplex Tableau will have ***m + n*** rows and ***m × n*** columns, plus *rhs*, with all constraint coefficients equal to **0** and **1**.

Example: m=4 sources, n=5 destinations:

C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	
1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	S
0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	S
0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	S
0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	S
1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	D
0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	D
0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	D
0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	D
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	D

m+n=9 constraints and m×n=20 variables

The **Transportation Simplex Method** uses a small "tableau",

with m+1 rows and n+1 columns:

	dstn	#1	dstn	#2	dstn	#3		dstn	#n	SUPPLY
source #1		C11		C12		C13			C1n	S1
source #2		C21		C22		C23			C2n	S2
source #3		C31		C32		C33			C3n	S3
source #m		Cm1		Cm2		Cm3			Cmn	Sm
DEMAND	D1		D2		D3			Dn		

The Simplex Method

- requires an ***initial basic feasible solution***
- at each iteration,
  - ✓computes ***reduced costs*** of nonbasic variables
  - ✓tests for ***optimality*** (stop if all reduced costs are nonnegative)
  - ✓selects variable to ***enter basis*** (with negative reduced cost)
  - ✓selects variable to ***leave basis*** (minimum ratio test)
  - ✓changes basis (***pivot***)

*The **Transportation Simplex Method** follows these same steps, but **without** pivoting in the ordinary simplex tableau.*

Obtaining an Initial Basic Feasible Solution

Commonly-used methods:

- Northwest-Corner Method
- Least-Cost Method
- Vogel's Approximation Method (**VAM**)
- Russell's Method

Some of these give better quality initial solutions (which often requires fewer iterations of the simplex method in optimization) than others.

Finding a starting basic feasible solution

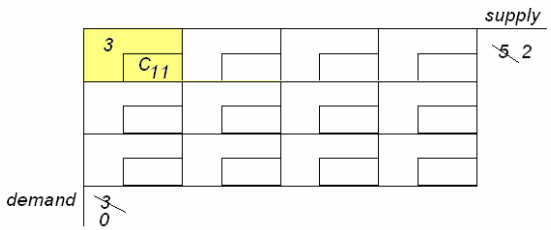
At each iteration,

- Select a cell *(i,j)* in the transportation tableau.
- Ship the maximum possible amount from source *i* to destination j, which is X_{ij} = minimum{S_i, D_j}.
- Reduce the available supply and demand:
  $S_i \leftarrow S_i - X_{ij}$, $D_j \leftarrow D_j - X_{ij}$
- Either S_i or D_j *(possibly both)* will now be zero. Remove it *(but only one of them if both are zero!)* from the transportation tableau.

The ***Northwest Corner*** method selects the cell in the ***upper left*** ("northwest") corner of the tableau (minus any rows & columns which have been removed) as the next cell to be assigned a shipment.

Example:

If we choose the "northwest corner" of the tableau, the most which could be shipped is 3 (the demand).



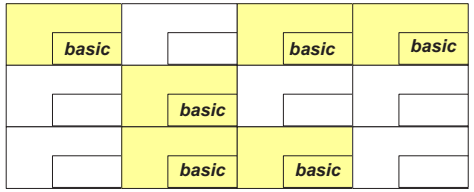
The Northwest Corner method ignores costs, and a basic feasible solution with a lower cost will generally be found by one of the other methods.

When *both* supply and demand are reduced to zero at some iteration, only the row ***or*** the column (not both) are removed. The next iteration will therefore result in an assignment of ***zero*** to the shipment. The basic solution which results is then **degenerate**.

Every basic feasible solution has the property that:

...including any single non-basic variable will form a "cycle" of shipments, alternately increasing and decreasing, which will yield feasible solutions.

This cycle of increases & decreases is necessary to keep the solution feasible.

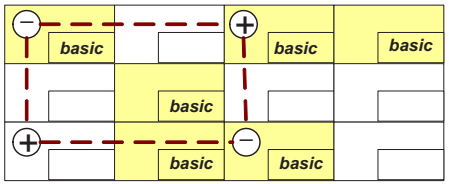


Example: m=3, n=4,

so that # basic variables

$= m + n - 1 = 6$

If we select the ***nonbasic*** variable X_{31} to be increased,

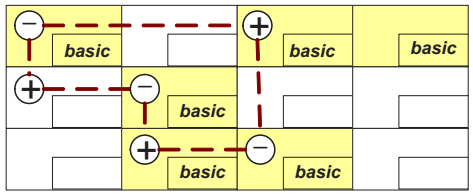


- *nonbasic* variable X_{31} increases by θ,
- basic variable X_{11} decreases by θ,
- basic variable X_{13} increases by θ,
- basic variable X_{33} decreases by θ,

a solution results in which

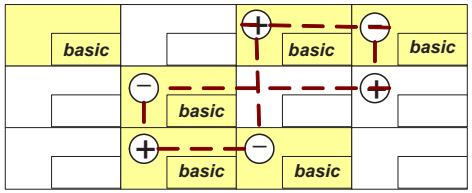
- the *total* shipments received from sources 1 & 3 are unchanged, and
- the *total* amount received by destinations 1 & 3 are unchanged.

If



- nonbasic variable X_{21} increases by θ,
- basic variable X_{11} decreases by θ,
- basic variable X_{13} increases by θ,
- basic variable X_{33} decreases by θ
- basic variable X_{32} increases by θ,
- basic variable X_{22} decreases by θ,

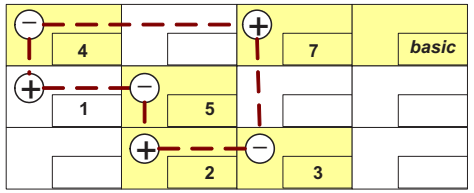
the ***total*** shipments ***from*** the sources and ***to*** the destinations is unchanged.



Likewise, if *nonbasic* variable X_{24} increases by θ, a sequence of decreases and increases around a cycle results in ***zero changes*** in ***total*** shipments from the sources and to the destinations.

Computing Reduced Cost of a Nonbasic Variable

(effect on total cost if a nonbasic variable is increased by 1)



Example:

If the nonbasic variable X_{21} is increased by 1, and the basic variables in the cycle are adjusted accordingly, the total change in the cost will be

Variable	X_{21}	X_{11}	X_{13}	X_{33}	X_{32}	X_{22}	**Total**
ΔX	+1	–1	+1	–1	+1	–1	**0**
Δcost	+1	–4	+7	–3	+2	–5	**–2**

The reduced cost of the nonbasic variable X_{21} is –2,
i.e., the total cost would decrease by 2 for every unit increase in X_{21}.

Choosing a Nonbasic Variable to Enter the Basis

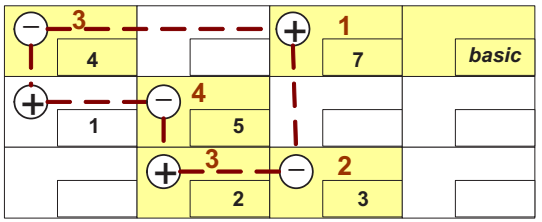
Any nonbasic variable with a negative reduced cost will improve the solution if it is entered into the basis.

(It does not need to be the variable with the smallest, i.e., the "most negative", reduced cost!)

Choosing a Basic Variable to Leave the Basis

As the selected nonbasic variable is increased, two or more of the basic variables will decrease.

When the ***first*** basic variable decreases to ***zero***, that variable must leave the basis.

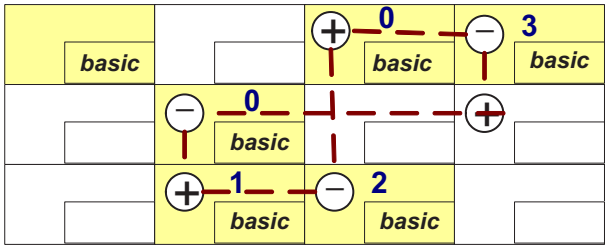


If ***more than one*** basic variable simultaneously reaches zero, ***one*** (only!) of these variables is arbitrarily selected to leave the basis.

Example: As X_{21} increases, the basic variables X_{11}, X_{22}, and X_{33} all decrease.
As soon as X_{21} has increased by **2** units (the smallest of the values of the decreasing variables), X_{33} reaches zero, and must leave the basis.

Changing the Basis ("pivot")

- A change of the basis requires that the values of the entering variable and the basic variables be adjusted as discussed during the choice of the variable to leave the basis.
- If ***more than one*** variable simultaneously reaches zero, only one is removed from the basis. Any other basic variable at value zero must remain in the basis, and this must be indicated in some way.
- A basic feasible solution with one or more basic variables equal to zero is called ***degenerate***.



"Stalling" Resulting from Degeneracy

Example: In the tableau shown, the solution is *degenerate* (X_{22} = **0** but ***basic***).

Any positive increase in the entering variable X_{24} would cause X_{22} to become negative!

Therefore, X_{24} *cannot increase…*

nevertheless, X_{24} should become *basic* and X_{22} *nonbasic*.

In the case of degeneracy, ***stalling*** may continue for more than one iteration, but eventually a basis change will be found in which the cells of the cycle which are decreased do not include a zero shipment.

The result then will be a basic feasible solution which is ***nondegenerate***.

Important property of the Transportation Problem

If the supplies and demands are ***integer***, then

- the ***initial*** basic feasible solution (e.g., from Northwest Corner method) will have only ***integer*** shipments,
- each basis change will result in a solution with ***integer*** shipments,

and so

the ***optimal*** basic solution will have only ***integer*** shipments!

This is true of many network flow LP models!