Facility Location Problems in the Plane

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Suppose that we wish to select the location of a single facility, anywhere in the plane, to serve a set of demand points.

Given, for each of demand points $j=1, 2, \ldots n$:

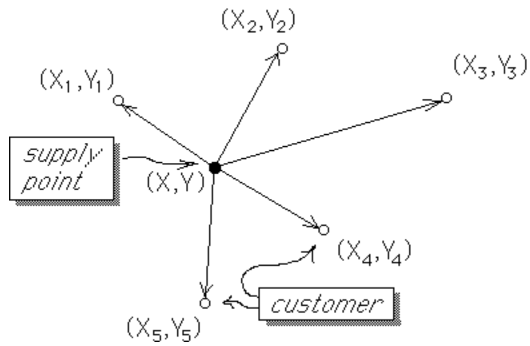
- (x_j, y_j) coordinates of the point
- β_j cost per unit volume per unit distance
- w_j volume of shipments per unit time

Find coordinates of the source facility, (x,y), which will minimize the total shipping cost per unit time:

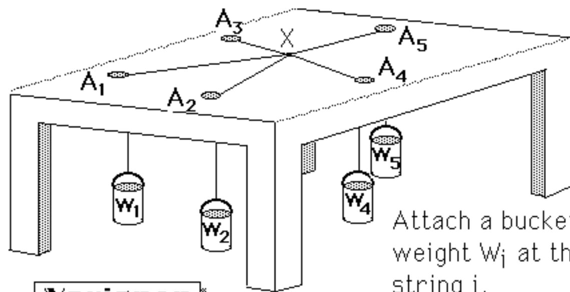
$$\text{Minimize } C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

assuming "straight-line", Euclidean distances!

Weber's Problem



Tie together in a knot ("X") n strings of equal length L



Attach a bucket with weight W_i at the end of string i.

Varignon Frame

Details

Theorem

The function:

$$C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

is convex in (x,y)

A *necessary* condition for (X^*,Y^*) to minimize

$$C(x,y) = \sum_{j=1}^{n} \beta_j w_j \sqrt{(x-x_j)^2 + (y-y_j)^2}$$

is

$$\begin{cases} \dfrac{\partial}{\partial X} C(X^*,Y^*) = 0 \\ \dfrac{\partial}{\partial Y} C(X^*,Y^*) = 0 \end{cases}$$

That is, (X^*,Y^*) should be a "stationary point" of the function C.

This condition yields the equations

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\beta_j w_j (X^*-x_j)}{\sqrt{(X^*-x_j)^2 + (Y^*-y_j)^2}} = 0 \\ \displaystyle\sum_{j=1}^{n} \frac{\beta_j w_j (Y^*-y_j)}{\sqrt{(X^*-x_j)^2 + (Y^*-y_j)^2}} = 0 \end{cases}$$

which, unfortunately, we cannot solve analytically for the values of X^* and Y^*!

For convenience, define a distance function for each j:

$$d_j(X,Y) = \sqrt{(X-x_j)^2 + (Y-y_j)^2}$$

Necessary conditions for optimality

$$\begin{cases} \displaystyle\sum_{j=1}^{n} \frac{\beta_j w_j (X^*-x_j)}{d_j(X^*,Y^*)} = 0 \\ \displaystyle\sum_{j=1}^{n} \frac{\beta_j w_j (Y^*-y_j)}{d_j(X^*,Y^*)} = 0 \end{cases}$$

Rearrange terms:

$$\begin{cases} X^* \sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)} = \sum_{j=1}^{n} \dfrac{\beta_j w_j x_j}{d_j(X^*,Y^*)} \\ Y^* \sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)} = \sum_{j=1}^{n} \dfrac{\beta_j w_j y_j}{d_j(X^*,Y^*)} \end{cases}$$

Necessary Conditions for the Optimality of (X^*, Y^*)

$$\begin{cases} X^* = \dfrac{\sum_{j=1}^{n} \dfrac{\beta_j w_j x_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)}} \\ Y^* = \dfrac{\sum_{j=1}^{n} \dfrac{\beta_j w_j y_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)}} \end{cases}$$

Note: X* and Y* actually appear on both sides of the equations!

We will use a "successive substitution" method using these equations to find X^* & Y^*

$$\begin{cases} X^* = \dfrac{\sum_{j=1}^{n} \dfrac{\beta_j w_j x_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)}} \\ Y^* = \dfrac{\sum_{j=1}^{n} \dfrac{\beta_j w_j y_j}{d_j(X^*,Y^*)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^*,Y^*)}} \end{cases}$$

Suppose, at iteration #k, we have an approximate solution (X^k, Y^k).
We obtain an improved approximate solution (X^{k+1}, Y^{k+1}) by

$$X^{k+1} = \frac{\sum_{j=1}^{n} \dfrac{\beta_j w_j x_j}{d_j(X^k,Y^k)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^k,Y^k)}} \quad \& \quad Y^{k+1} = \frac{\sum_{j=1}^{n} \dfrac{\beta_j w_j y_j}{d_j(X^k,Y^k)}}{\sum_{j=1}^{n} \dfrac{\beta_j w_j}{d_j(X^k,Y^k)}}$$

Weiszfeld Algorithm

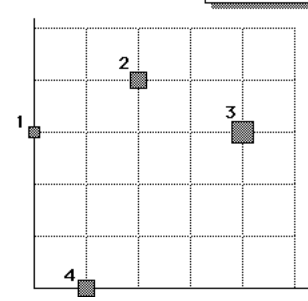
Starting with an initial "guess" (X^0, Y^0), we will generate a sequence of approximate solutions, (X^1,Y^1), (X^2,Y^2), (X^3,Y^3), ... which converge to the optimal facility location (X^*,Y^*).

We terminate the method when two successive approximate solutions are "close enough", i.e.,

$$\left|X^{k+1} - X^k\right| + \left|Y^{k+1} - Y^k\right| < \varepsilon \approx 0$$

Example

Customer	1	2	3	4
Location	(0,3)	(2,4)	(4,3)	(1,0)
Rqmt.(Ton/wk)	1	2	3	2



Cost/ton-mile is same for all customers

Where should a supply facility be located so that total shipping cost per week is minimized?

Starting Point

A good starting point is the centroid, i.e., the weighted average of the customer coordinates.

$$X^0 = \frac{\sum_{j=1}^{4} \beta_j w_j X_j}{\sum_{j=1}^{4} \beta_j w_j} \quad \& \quad Y^0 = \frac{\sum_{j=1}^{4} \beta_j w_j Y_j}{\sum_{j=1}^{4} \beta_j w_j}$$

Customer	1	2	3	4
Location	(0,3)	(2,4)	(4,3)	(1,0)
Rqmt.	1	2	3	2

$\beta_j = 1 \ \forall j$

$$X^0 = \frac{\sum_{j=1}^{4} \beta_j w_j X_j}{\sum_{j=1}^{4} \beta_j w_j} = \frac{1\times0+2\times2+3\times4+2\times1}{1+2+3+2} = 2.25$$

$$Y^0 = \frac{\sum_{j=1}^{4} \beta_j w_j Y_j}{\sum_{j=1}^{4} \beta_j w_j} = \frac{1\times3+2\times4+3\times3+2\times1}{1+2+3+2} = 2.5$$

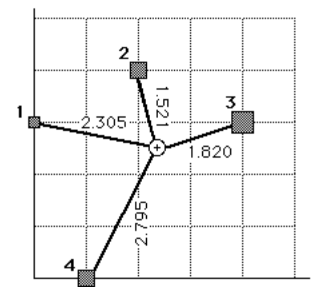
Now compute distance from (X^0, Y^0) to each customer:

$$d_1 = \sqrt{\left(\frac{9}{4}-0\right)^2 + \left(\frac{5}{2}-3\right)^2} \approx 2.305$$

$$d_2 = \sqrt{\left(\frac{9}{4}-2\right)^2 + \left(\frac{5}{2}-4\right)^2} \approx 1.521$$

$$d_3 = \sqrt{\left(\frac{9}{4}-4\right)^2 + \left(\frac{5}{2}-3\right)^2} \approx 1.820$$

$$d_4 = \sqrt{\left(\frac{9}{4}-1\right)^2 + \left(\frac{5}{2}-0\right)^2} \approx 2.795$$



Shipping Cost:

1(2.305) + 2(1.521) + 3(1.82) + 2(2.795) = 16.397

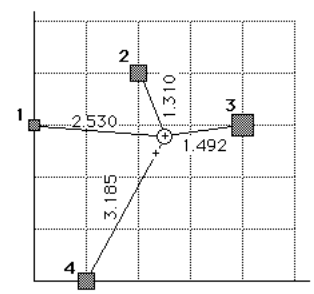
Apply our successive substitution method to (we hope!) obtain a better approximate solution:

$$X^1 = \frac{\sum_j \frac{w_j X_j}{d_j}}{\sum_j \frac{w_j}{d_j}} = \frac{\frac{1\times \mathbf{0}}{d_1}+\frac{2\times 2}{d_2}+\frac{3\times 4}{d_3}+\frac{2\times 1}{d_4}}{\frac{1}{d_1}+\frac{2}{d_2}+\frac{3}{d_3}+\frac{2}{d_4}} \approx \frac{10.373}{4.113} = 2.522$$

$$Y^1 = \frac{\sum_j \frac{w_j Y_j}{d_j}}{\sum_j \frac{w_j}{d_j}} = \frac{\frac{1\times 3}{d_1}+\frac{2\times 4}{d_2}+\frac{3\times 3}{d_3}+\frac{2\times 0}{d_4}}{\frac{1}{d_1}+\frac{2}{d_2}+\frac{3}{d_3}+\frac{2}{d_4}} \approx \frac{11.506}{4.113} = 2.798$$

distance to customer j

$$d_j = \sqrt{[2.522-X_j]^2 + [2.798-Y_j]^2}$$



$d_1 = 2.530$

$d_2 = 1.310$

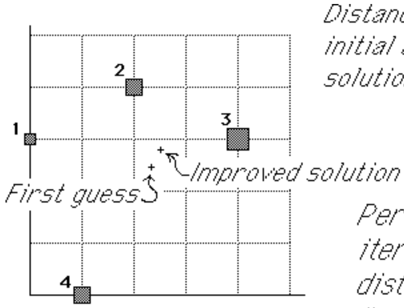
$d_3 = 1.492$

$d_4 = 3.185$

Shipping cost

1(2.53)+2(1.31)+3(1.492) + 2(3.185) = 15.996 < 16.397

reduction of 2.4%



Distance between initial and improved solution:

0.421

Perform additional iterations, until distance moved is "sufficiently small"

Iteration # 1

Facility location at X= 2.25, Y= 2.5

Distances to demand pts:

i	1	2	3	4
D[i]	2.30489	1.52069	1.82003	2.79508
WT[i]×D[i]	2.30489	3.04138	5.46008	5.59017

Total cost is 16.3965

New location is at X= 2.41659, Y= 2.79785

Rectilinear distance moved is 0.464436

Iteration # 2

Facility location at X= 2.41659, Y= 2.79785

Distances to demand pts:

i	1	2	3	4
D[i]	2.42503	1.27229	1.59626	3.13603
WT[i]×D[i]	2.42503	2.54457	4.78879	6.27206

Total cost is 16.0305

New location is at X= 2.51012, Y= 2.92419

Rectilinear distance moved is 0.21987

Iteration # 3

Facility location at X= 2.51012, Y= 2.92419

Distances to demand pts:

i	1	2	3	4
D[i]	2.51126	1.19063	1.49181	3.2911
WT[i]×D[i]	2.51126	2.38126	4.47542	6.5822

Total cost is 15.9501

New location is at X= 2.55738, Y= 2.96949

Rectilinear distance moved is 0.0925647

Iteration # 4

Facility location at X= 2.55738, Y= 2.96949

Distances to demand pts:

i	1	2	3	4
D[i]	2.55757	1.1716	1.44294	3.3531
WT[i]×D[i]	2.55757	2.34319	4.32881	6.7062

Total cost is 15.9358

New location is at X= 2.58231, Y= 2.98276

Rectilinear distance moved is 0.0381954

Iteration # 5

Facility location at X= 2.58231, Y= 2.98276

Distances to demand pts:

i	1	2	3	4
D[i]	2.58237	1.17212	1.4178	3.37647
WT[i]×D[i]	2.58237	2.34424	4.25339	6.75294

Total cost is 15.9329

New location is at X= 2.59667, Y= 2.98528

Rectilinear distance moved is 0.0168786

Iteration # 6

Facility location at X= 2.59667, Y= 2.98528

Distances to demand pts:

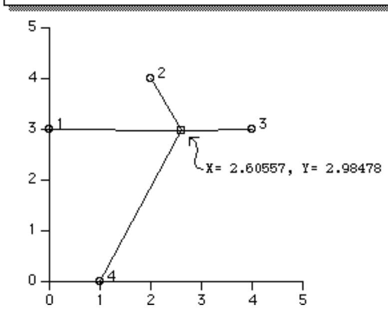
i	1	2	3	4
D[i]	2.59671	1.17715	1.40341	3.38544
WT[i]×D[i]	2.59671	2.35429	4.21023	6.77089

Total cost is 15.9321

New location is at X= 2.60557, Y= 2.98478

Rectilinear distance moved is 0.0094046 < 0.01 (stopping criterion)

The Optimal Location for the Supply Facility



Path Followed by the Successive Substitution Method