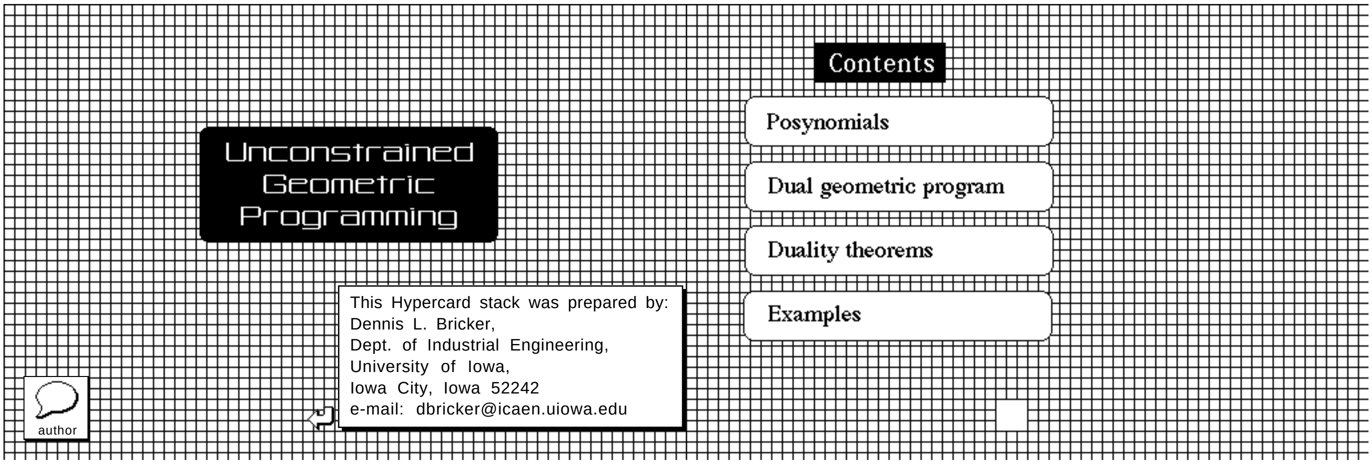

Unconstrained Geometric Programming

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A function $g(t)$ defined for all $t = (t_1, t_2, \ldots t_m)$ in R^m with $t_i > 0$ for all $i=1, 2, \ldots m$ is called a ***posynomial*** if it is of the form

$$g(t) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

where the c_i's are ***positive*** constants, and the exponents a_{ij}'s are real numbers

Examples of posynomials

$$g(t) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

$$\frac{2z\sqrt{y}}{x} \qquad \textit{i.e.,}\quad 2x^{-2}\,y^{1/2}\,z$$

$$\frac{\sqrt{x_1}}{x_2} + 3x_2^2\,x_3 \qquad \textit{i.e.,}\quad x_1^{0.5}\,x_2^{-1} + 3x_2^2\,x_3$$

Note that not all posynomials are polynomials, and not all polynomials are posynomials!

PRIMAL GEOMETRIC PROGRAM

(unconstrained case)

Minimize the posynomial $g(t) = \sum_{i=1}^{n} c_i \prod_{j=1}^{m} t_j^{a_{ij}}$

subject to $t_j > 0$ for $j=1,2, \ldots m$

Rewrite $g(t)$:

$$g(t) = \sum_{i=1}^{n} \delta_i \left(\frac{c_i \prod_{j=1}^{m} t_j^{a_{ij}}}{\delta_i} \right) \quad \text{where } \delta_i > 0 \quad \text{and } \sum_{i=1}^{n} \delta_i = 1$$

Apply the Arithmetic-Geometric Mean Inequality:

$$g(t) \geq \prod_{i=1}^{n} \left(\frac{c_i \prod_{j=1}^{m} t_j^{a_{ij}}}{\delta_i} \right)^{\delta_i} = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \left(\prod_{i=1}^{n} \prod_{j=1}^{m} t_j^{a_{ij}\delta_i} \right)$$

$$g(t) \geq \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \left(\prod_{j=1}^{m} \prod_{i=1}^{n} t_j^{a_{ij}\delta_i} \right) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{j=1}^{m} t_j^{\sum_{i=1}^{n} a_{ij}\delta_i}$$

We would like this lower bound to be not dependent on the variables t_j, $j=1, 2, \ldots m$

This will be so if their exponents are zero:

$$\sum_{i=1}^{n} a_{ij}\,\delta_i = 0, \quad j=1,2,\ldots\ m$$

Under the restrictions that

$$\begin{cases} \sum_{i=1}^{n} \delta_i = 1 & \textit{Normality condition} \\ \sum_{i=1}^{n} a_{ij}\,\delta_i = 0, \quad j=1,2,\ldots\ m & \textit{Orthogonality conditions (one per primal variable)} \\ \delta_i > 0, \quad i=1, 2, \ldots\ n & \end{cases}$$

then

$$g(t) \geq \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}$$

Define the function $v(\delta) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}$ *dual function*

then for all $t > 0$ and nonnegative δ satisfying the normality & orthogonality conditions,

$$\boxed{\sum_{i=1}^{n} c_i \prod_{j=1}^{m} t_j^{a_{ij}} = g(t) \geq v(\delta) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}}$$

the Primal - Dual Inequality

Dual Geometric Program

DGP: Maximize $v(\delta) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i}$
subject to

$$\begin{cases} \sum_{i=1}^{n} \delta_i = 1 & \textit{Normality condition} \\ \sum_{i=1}^{n} a_{ij}\,\delta_i = 0,\ j=1,2,\ldots\ m & \textit{Orthogonality conditions} \\ \delta_i > 0\ ,\ i=1,\,2,\,\ldots\ n & \end{cases}$$

Weak Duality Theorem

If t^* solves the primal geometric program (GP) and δ^* solves the dual geometric program (DGP)

then

$$g(t^*) \geq v(\delta^*)$$

Proof: The Primal-Dual Inequality

(We will next show that the above inequality is tight, i.e., that the strong duality property holds.)

Strong Duality Theorem

If $t^* = (t_1^*, t_2^*, \ldots\ t_m^*)$ solves the primal GP,

then the dual GP is consistent, and the vector

$\delta^* = (\delta_1^*, \delta_2^*, \ldots\ \delta_n^*)$ defined by $\delta_i^* = \dfrac{c_i \prod_{j=1}^{m} t_j^{*\,a_{ij}}}{g(t^*)}$

is a solution for the dual GP, and $g(t^*) = v(\delta^*)$.

$$\delta_i^* = \frac{c_i \prod_{j=1}^{m} t_j^{*\,a_{ij}}}{g(t^*)}$$

term # i of the primal objective

optimal value of posynomial

That is, the optimal dual variable δ_i^*, associated with term i of the primal objective function, is simply the *fraction of the optimal cost which is contributed by that term!*

Proof

Proof

Notation $\quad g(t) = \sum_{i=1}^{n} u_i(t)$ **where** $u_i(t) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}$

Outline of Proof: If we define $\delta_i^* = \dfrac{u_i(t^*)}{g(t^*)}$

where t^* minimizes $g(t)$, then

δ^* is feasible in the dual, and $g(t^*) = v(\delta^*)$.

By the weak duality theorem, δ^* solves the dual GP.

$$g(t) = \sum_{i=1}^{n} u_i(t) \text{ where } u_i(t) = c_i \prod_{j=1}^{m} t_j^{a_{ij}}$$

If t^* minimizes $g(t)$, then t^* satisfies (for $k=1, \ldots m$)

$$0 = \frac{\partial}{\partial t_k} g(t) = \sum_{i=1}^{n} \frac{\partial}{\partial t_k} u_i(t) = \sum_{i=1}^{n} \frac{\partial}{\partial t_k} c_i \prod_{j=1}^{m} t_j^{a_{ij}} = \sum_{i=1}^{n} c_i a_{ik} t_k^{a_{ik}-1} \prod_{j \neq k} t_j^{a_{ij}}$$

Multiply both sides by t_k:

$$t_k \times 0 = t_k \sum_{i=1}^{n} c_i a_{ik} t_k^{a_{ik}-1} \prod_{j \neq k} t_j^{a_{ij}} = \sum_{i=1}^{n} a_{ik} c_i \prod_{j=1}^{m} t_j^{a_{ij}} = \sum_{i=1}^{n} a_{ik} u_i(t)$$

Therefore, if t^* minimizes $g(t)$, then t^* satisfies

$$0 = \sum_{i=1}^{n} a_{ik} u_i(t^*) \Longrightarrow \quad 0 = \sum_{i=1}^{n} a_{ik} \frac{u_i(t^*)}{g(t^*)}\ ,\ k=1,\ \ldots m$$

If we let $\delta_i^* = \dfrac{u_i(t^*)}{g(t^*)}$, $i=1,\,2,\,\ldots\ n$

then $\delta^* = (\delta_1^*, \delta_2^*, \ldots\ \delta_n^*)$ satisfies

$$0 = \sum_{i=1}^{n} a_{ik} \delta_i^*\ ,\ k=1,\ \ldots m$$

(orthogonality conditions are satisfied!)

Also, $\delta_i^* = \dfrac{u_i(t^*)}{g(t^*)} > 0$

and

$$\sum_{i=1}^{n} \delta_i^* = \sum_{i=1}^{n} \frac{u_i(t^*)}{g(t^*)} = \frac{\sum_{i=1}^{n} u_i(t^*)}{g(t^*)} = 1$$

(normality condition is satisfied!)

Therefore, δ^* is feasible in the dual GP.

Also, $g(t^*) = g(t^*)^{\delta_1 + \dots + \delta_n} = g(t^*)^{\delta_1}\, g(t^*)^{\delta_2} \cdots g(t^*)^{\delta_n}$

$$= \left[\frac{u_1(t^*)}{\delta_1}\right]^{\delta_1} \left[\frac{u_2(t^*)}{\delta_2}\right]^{\delta_2} \cdots \left[\frac{u_n(t^*)}{\delta_n}\right]^{\delta_n}$$

$$= \left[\frac{c_1}{\delta_1}\right]^{\delta_1} \left[\frac{c_2}{\delta_2}\right]^{\delta_2} \cdots \left[\frac{c_n}{\delta_n}\right]^{\delta_n} \left[\prod_{j=1}^{m} t_j^{*\,a_{ij}}\right]^{\delta_1} \left[\prod_{j=1}^{m} t_j^{*\,a_{ij}}\right]^{\delta_2} \cdots \left[\prod_{j=1}^{m} t_j^{*\,a_{ij}}\right]^{\delta_n}$$

$$g(t^*) = \left[\frac{c_1}{\delta_1}\right]^{\delta_1} \left[\frac{c_2}{\delta_2}\right]^{\delta_2} \cdots \left[\frac{c_n}{\delta_n}\right]^{\delta_n} \prod_{j=1}^{m} t_j^{*\,\sum_{i=1}^{n} a_{ij}\delta_i}$$

$$= \left[\frac{c_1}{\delta_1}\right]^{\delta_1} \left[\frac{c_2}{\delta_2}\right]^{\delta_2} \cdots \left[\frac{c_n}{\delta_n}\right]^{\delta_n} \prod_{j=1}^{m} t_j^{*\,0}$$

$$= \left[\frac{c_1}{\delta_1}\right]^{\delta_1} \left[\frac{c_2}{\delta_2}\right]^{\delta_2} \cdots \left[\frac{c_n}{\delta_n}\right]^{\delta_n} = v(\delta^*)$$

That is, if t^* is optimal in the primal GP, then δ^* is feasible and optimal in DGP, where

$$\delta_i^* = \frac{c_i \prod_{j=1}^{m} t_j^{*\,a_{ij}}}{g(t^*)}$$

and

$$g(t^*) = v(\delta^*)$$

Q.E.D.

"Degrees of Difficulty"

of a geometric program is

$$\underbrace{(\#\text{ of terms})}_{\text{number of dual variables}} - \underbrace{(\ 1 + \#\text{ of variables})}_{\text{number of dual equality constraints}}$$

(normality + orthogonality)

Examples

- Minimize $C_1x^3 + \frac{C_2}{x}$
- Minimize $1/x$ s.t. $x>0$
- Minimize x s.t. $x>0$
- Gravel Box Design
- Gravel Box with Runners
- Furnace
- Gas Transmission Line

Example

Minimize $1/x$ s.t. $x>0$

The minimum is not attained at any finite value of x

There is a single term, so that the dual program is

DGP:

$$\begin{aligned} &\text{Maximize } \left(\frac{1}{\delta}\right)^{\delta} \\ &\text{subject to } \quad \delta = 1 \\ &\qquad\qquad\quad -\delta = 0 \\ &\qquad\qquad\quad\ \ \delta \geq 0 \end{aligned}$$

← *normality condition*

← *orthogonality condition*

There is no feasible solution of th dual geometric program!

Example

Minimize x s.t. $x>0$

The minimum is not attained at any positive value of x

There is a single term, so that the dual program is

DGP:

$$\begin{aligned} &\text{Maximize } \left(\frac{1}{\delta}\right)^{\delta} \\ &\text{subject to } \quad \delta = 1 \\ &\qquad\qquad\quad\ \ \delta = 0 \\ &\qquad\qquad\quad\ \ \delta \geq 0 \end{aligned}$$

← *normality condition*

← *orthogonality condition*

There is no feasible solution of th dual geometric program!

Example

GP: Minimize $f(x) = C_1x^3 + \frac{C_2}{x}$, where $C_1>0$ & $C_2>0$

Define two dual variables, one per term of the posynomial:

$$C_1x^3 \longleftrightarrow \delta_1$$

$$\frac{C_2}{x} = C_2x^{-1} \longleftrightarrow \delta_2$$

$$\begin{cases} \delta_1 + \delta_2 = 1 & \leftarrow \textit{Normality constraint} \\ 3\,\delta_1 - \delta_2 = 0 & \leftarrow \textit{Orthogonality constraint} \\ \delta_1 > 0 \ \& \ \delta_2 > 0 \end{cases}$$

The dual objective:

$$v(\delta) = \prod_{i=1}^{n}\left(\frac{C_i}{\delta_i}\right)^{\delta_i} = \left(\frac{C_1}{\delta_1}\right)^{\delta_1}\left(\frac{C_2}{\delta_2}\right)^{\delta_2}$$

DGP: Maximize $\left(\frac{C_1}{\delta_1}\right)^{\delta_1}\left(\frac{C_2}{\delta_2}\right)^{\delta_2}$

subject to

$$\begin{cases} \delta_1 + \delta_2 = 1 & \leftarrow \textit{Normality constraint} \\ 3\,\delta_1 - \delta_2 = 0 & \leftarrow \textit{Orthogonality constraint} \\ \delta_1 > 0 \ \& \ \delta_2 > 0 \end{cases}$$

The constraints of DGP (2 equations with 2 variables) have, in this example, a *unique feasible solution*:

$$\begin{cases} \delta_1 + \delta_2 = 1 \\ 3\,\delta_1 - \delta_2 = 0 \\ \delta_1 > 0 \ \& \ \delta_2 > 0 \end{cases} \Rightarrow \begin{matrix} \delta_1 = 1/4 \\ \delta_2 = 3/4 \end{matrix}$$

$$v(\delta) = \left(\frac{C_1}{\delta_1}\right)^{\delta_1}\left(\frac{C_2}{\delta_2}\right)^{\delta_2} = \left(\frac{C_1}{1/4}\right)^{1/4}\left(\frac{C_2}{3/4}\right)^{3/4} = 4\left(\frac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4}$$

Computation of the Primal Optimal Solution

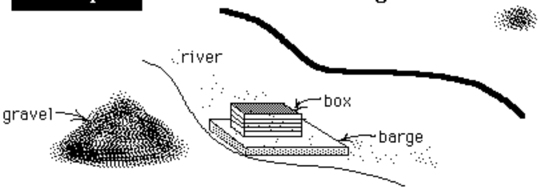
$$\delta_1 = 1/4 = \frac{C_1x^{*3}}{f(x^*)} \qquad \delta_2 = 3/4 = \frac{\frac{C_2}{x^*}}{f(x^*)}$$

Since $f(x^*) = v(\delta^*) = 4\left(\frac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4}$

$$\frac{C_2}{x^*} = 3/4\,(4)\left(\frac{1}{3}\right)^{3/4} C_1^{1/4} C_2^{3/4} \Rightarrow x^* = \left(\frac{1}{3}\right)^{1/4} C_1^{-1/4} C_2^{1/4}$$

In this example, DGP had a unique feasible solution... such is not the case in general, so that one must actually solve a maximization problem in DGP!

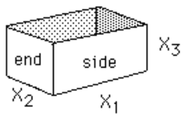
Example: Gravel Box Design



400 m of gravel is to be ferried across a river on a barge. A box (with open top) is to be built for this purpose. After all the gravel has been ferried, the box is to be discarded.

Costs:
- ***transport*** 10¢ per round trip of barge
- ***materials*** sides & bottom of box: $10/m
  - ends of box: $20/m

Decision Variables



x_1 = length (m)
x_2 = width (m)
x_3 = height (m)

Volume

$x_1x_2x_3$ (m^3)

Areas

end: x_2x_3 (m^2)
side: x_1x_3 (m^2)
bottom: x_1x_2 (m^2)

Cost function

Transport cost: $\left(0.10\ \$/\text{trip}\right)\dfrac{400\ m^3}{x_1x_2x_3\ m^3/\text{trip}}$

Materials cost:

$$\begin{cases} \text{ends of box: } 2\left(20\ \$/m^2\right)x_2x_3\ m^2 \\ \text{sides of box: } 2\left(10\ \$/m^2\right)x_1x_3\ m^2 \\ \text{bottom: } 2\left(10\ \$/m^2\right)x_1x_2\ m^2 \end{cases}$$

total cost

$$f(x) = \underbrace{40x_1^{-1}x_2^{-1}x_3^{-1}}_{\text{transport}} + \underbrace{40x_2x_3}_{\text{ends}} + \underbrace{20x_1x_3}_{\text{sides}} + \underbrace{10x_1x_2}_{\text{bottom}}$$

a posynomial!

Define a "weight", i.e., dual variable, for each term of the cost function:

$$\underbrace{\delta_1 + \delta_2 + \delta_3 + \delta_4 = 1}_{\text{normality constraint}}, \quad \delta_i > 0,\ i=1,2,3,4$$

In addition, there will be an "orthogonality constraint" for each of the (primal) variables.

cost: $\underbrace{40x_1^{-1}x_2^{-1}x_3^{-1}}_{\text{transport}} + \underbrace{40x_2x_3}_{\text{ends}} + \underbrace{20x_1x_3}_{\text{sides}} + \underbrace{10x_1x_2}_{\text{bottom}}$

$$= 40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1^0x_2^1x_3^1 + 20x_1^1x_2^0x_3^1 + 10x_1^1x_2^1x_3^0$$

The orthogonality constraint corresponding to x_j *is*

$$-\delta_1 + \delta_3 + \delta_4 = 0$$

(exponent of x_j *in term i is the coefficient of* δ_i*)*

$$\boxed{\sum_{i=1}^{n} a_{ij}\,\delta_i = 0}$$

$$\boxed{40x_1^{-1}x_2^{-1}x_3^{-1} + 40x_1^0x_2^1x_3^1 + 20x_1^1x_2^0x_3^1 + 10x_1^1x_2^1x_3^0}$$

exponent matrix:

$$A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{cases} -\delta_1 + \delta_3 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

orthogonality constraints

DGP

$$\text{maximize } v(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{40}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{10}{\delta_4}\right)^{\delta_4}$$

$$\begin{cases} \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1 \\ -\delta_1 + \delta_3 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

$$\delta_i > 0,\ i=1,2,3,4$$

In this problem, again DGP has an equal number of variables and equations, with a unique feasible solution:

$$\begin{cases} \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1, \\ -\delta_1 + \delta_3 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

Gauss-Jordan elimination yields

$\delta_1 = 0.4$
$\delta_2 = 0.2$
$\delta_3 = 0.2$
$\delta_4 = 0.2$

cost: $\underbrace{40x_1^{-1}x_2^{-1}x_3^{-1}}_{\text{transport}} + \underbrace{40x_2x_3}_{\text{ends}} + \underbrace{20x_1x_3}_{\text{sides}} + \underbrace{10x_1x_2}_{\text{bottom}}$

$\delta_1 = 0.4$
$\delta_2 = 0.2$
$\delta_3 = 0.2$
$\delta_4 = 0.2$

We immediately have the result that

transportation cost = 40% of total cost
cost of ends = 20% of total cost
cost of sides = 20% of total cost
cost of bottom = 20% of total cost

This is independent of the cost coefficients!
(e.g., 10¢/trip, etc.)

Computation of the optimal primal variables

$$\delta_1^* = 0.4 = \left(\frac{40x_1^{-1}x_2^{-1}x_3^{-1}}{f(x^*)}\right)$$

$$\delta_2^* = 0.2 = \left(\frac{40x_2x_3}{f(x^*)}\right)$$

$$\delta_3^* = 0.2 = \left(\frac{20x_1x_3}{f(x^*)}\right)$$

$$\delta_4^* = 0.2 = \left(\frac{10x_1x_2}{f(x^*)}\right)$$

At the optimum solutions of the primal & dual, $f(x^*) = v(\delta^*)$

where

$$v(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{40}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{10}{\delta_4}\right)^{\delta_4}$$

$$v(\delta^*) = 100$$

Solve for x:

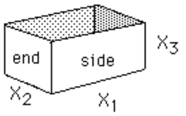
$$\begin{cases} 40x_1^{-1}x_2^{-1}x_3^{-1} = (0.4)(100) = 40 \\ 40x_2x_3 = (0.2)(100) = 20 \\ 20x_1x_3 = (0.2)(100) = 20 \\ 10x_1x_2 = (0.2)(100) = 20 \end{cases}$$

This can be done by taking logs of both sides, to get a linear system in the logarithms of x_i

$$\begin{cases} 40x_1^{-1}x_2^{-1}x_3^{-1} = 40 \\ 40x_2x_3 = 20 \\ 20x_1x_3 = 20 \\ 10x_1x_2 = 20 \end{cases} \Rightarrow \begin{cases} -z_1 - z_2 - z_3 = 0 \\ z_2 + z_3 = -\ln 2 \\ z_1 + z_3 = 0 \\ z_1 + z_2 = -\ln 2 \end{cases} \quad \boxed{z_i \equiv \ln x_i}$$

$$\Rightarrow \begin{cases} z_1 = \ln 2 \\ z_2 = 0 = \ln 1 \\ z_3 = -\ln 2 \end{cases} \Rightarrow \boxed{\begin{array}{l} x_1 = 2 \\ x_2 = 1 \\ x_3 = \tfrac{1}{2} \end{array}}$$

Decision Variables



$$\begin{cases} x_1 = \text{length} = 2 \text{ m} \\ x_2 = \text{width} = 1 \text{ m} \\ x_3 = \text{height} = \tfrac{1}{2} \text{ m} \end{cases}$$

Volume

$x_1x_2x_3 = 1 \text{ m}^3$

Areas

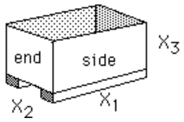
end: $x_2x_3 = \tfrac{1}{2} \text{ m}^2$
side: $x_1x_3 = 1 \text{ m}^2$
bottom: $x_1x_2 = 2 \text{ m}^2$

Cost

\$100 ⏎

Modification of original problem:

To ease sliding the box onto the barge, while eliminating wear & tear on the bottom, "runners" are to be placed on the bottom, along the length.
Cost of materials for runners: \$2.50/meter



Cost of runners:

$2(2.50)x_1 = 5x_1$

⏎

$$\text{cost: } \underbrace{40x_1^{-1}x_2^{-1}x_3^{-1}}_{transport} + \underbrace{40x_2x_3}_{ends} + \underbrace{20x_1x_3}_{sides} + \underbrace{10x_1x_2}_{bottom} + \underbrace{5x_1}_{runners}$$

This introduces a new dual variable, δ_5 *:*

$$\begin{cases} \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \\ -\delta_1 + \delta_3 + \delta_4 + \delta_5 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

Now there are 5 variables, but only 4 equations!

DGP The dual geometric program

$$\text{maximize } v(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{40}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{10}{\delta_4}\right)^{\delta_4}\left(\frac{5}{\delta_5}\right)^{\delta_5}$$

$$\begin{cases} \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \\ -\delta_1 + \delta_3 + \delta_4 + \delta_5 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

$\delta_i > 0,\ i = 1, 2, 3, 4, 5$

$$\begin{cases} \delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 = 1 \\ -\delta_1 + \delta_3 + \delta_4 + \delta_5 = 0 \\ -\delta_1 + \delta_2 + \delta_4 = 0 \\ -\delta_1 + \delta_2 + \delta_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} \begin{array}{ccccc} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0.2 & 0.4 \\ 0 & 1 & 0 & 0 & -0.4 & 0.2 \\ 0 & 0 & 1 & 0 & 0.6 & 0.2 \\ 0 & 0 & 0 & 1 & 0.6 & 0.2 \end{array}\right] \end{array}$$

Gauss-Jordan Elimination

$$\Rightarrow \begin{cases} \delta_1 = 0.4 - 0.2\,\delta_5 \\ \delta_2 = 0.2 + 0.4\,\delta_5 \\ \delta_3 = 0.2 - 0.6\,\delta_5 \\ \delta_4 = 0.2 - 0.6\,\delta_5 \end{cases}$$

$$v(\delta) = \left(\frac{40}{\delta_1}\right)^{\delta_1}\left(\frac{40}{\delta_2}\right)^{\delta_2}\left(\frac{20}{\delta_3}\right)^{\delta_3}\left(\frac{10}{\delta_4}\right)^{\delta_4}\left(\frac{5}{\delta_5}\right)^{\delta_5} \ \& \ \begin{cases} \delta_1 = 0.4 - 0.2\,\delta_5 \\ \delta_2 = 0.2 + 0.4\,\delta_5 \\ \delta_3 = 0.2 - 0.6\,\delta_5 \\ \delta_4 = 0.2 - 0.6\,\delta_5 \end{cases}$$

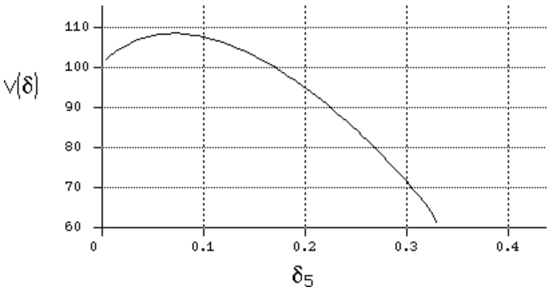
$$v(\delta) = \left(\frac{40}{.4-.2\delta_5}\right)^{.4-.2\delta_5}\left(\frac{40}{.2+.4\delta_5}\right)^{.2+.4\delta_5}\left(\frac{20}{.2-.6\delta_5}\right)^{.2-.6\delta_5}\left(\frac{10}{.2-.6\delta_5}\right)^{.2-.6\delta_5}\left(\frac{5}{\delta_5}\right)^{\delta_5}$$

We now have a function of a ***single variable*** *to be maximized, using, for example, golden-section search!*

What is the initial interval bounding the optimum?

$$\begin{cases} \delta_1 = 0.4 - 0.2\,\delta_5 \geq 0 \\ \delta_2 = 0.2 + 0.4\,\delta_5 \geq 0 \\ \delta_3 = 0.2 - 0.6\,\delta_5 \geq 0 \\ \delta_4 = 0.2 - 0.6\,\delta_5 \geq 0 \end{cases} \qquad \begin{cases} \delta_5 \leq 2 \\ \delta_5 \geq -.5 \\ \delta_5 \leq \tfrac{1}{3} \\ \delta_5 \leq \tfrac{1}{3} \end{cases}$$

$$\boxed{0 \leq \delta_5 \leq \tfrac{1}{3}}$$



δ_5	$v(\delta)$
0.005	102.012
0.01	103.334
0.015	104.388
0.02	105.261
0.025	105.995
0.03	106.615
0.035	107.135
0.04	107.567
0.045	107.92
0.05	108.201
0.055	108.414
0.06	108.564
0.065	108.655
0.07	108.69
0.075	108.672
0.08	108.602
0.085	108.484
0.09	108.318
0.095	108.108
0.1	107.853
0.105	107.557
0.11	107.219

δ_5	$v(\delta)$
0.115	106.842
0.12	106.427
0.125	105.974
0.13	105.485
0.135	104.96
0.14	104.401
0.145	103.809
0.15	103.183
0.155	102.526
0.16	101.837
0.165	101.118
0.17	100.369
0.175	99.5913
0.18	98.7847
0.185	97.9502
0.19	97.0883
0.195	96.1995
0.2	95.2843
0.205	94.3431
0.21	93.3765
0.215	92.3847
0.22	91.3681

$$\delta_5^* = 0.07 \pm 0.005$$

$$v(\delta^*) \doteq 108.69$$

The economic model for the annual cost of a furnace in which a slag-metal reaction is to be conducted is:

$$C(L,T) = 1\times10^{13}/L^3T^2 + 100L^2 + 5\times10^{-11}L^2T^4$$

where

L = characteristic length of furnace (feet)

T = temperature (^{0} Kelvin)

Find the minimum cost and the optimal L & T.

The annual costs of a gas transmission line are

$$C(L,D,F) = 4.55\times10^5\left(\frac{L^{1/2}}{F^{0.387}D^{2/3}}\right) + 3.69\times10^4 D + \frac{6.57\times10^6}{L} + 7.72\times10^8\frac{F}{L}$$

where L = pipe length between compressors (ft)

D = pipe diameter (inches)

$F = r^{0.219} - 1$

r = ratio of inlet to outlet pressure