Mathematical Programming Models of TSP

This Hypercard stack was prepared by:
Dennis L. Bricker,
Dept. of Industrial Engineering,
University of Iowa,
Iowa City, Iowa 52242
e-mail: dbricker@icaen.uiowa.edu

author

Minimize $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} C_{ij}X_{ij}$
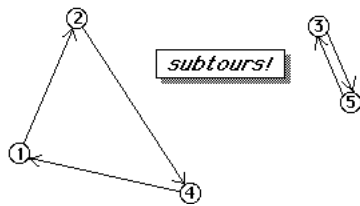
subject to

$\sum\limits_{j=1}^{n} X_{ij} = 1$ for i=1, …n   *one city must follow city i*

$\sum\limits_{i=1}^{n} X_{ij} = 1$ for j=1, …n   *one city must precede city j*

$X_{ij} \, \varepsilon \, \{0,1\}$ for all i,j

These are the constraints of the assignment problem!

⇦ ⇨

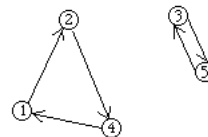**Not all feasible solutions of the assignment problem (AP) are TSP tours!**

subtours!

What constraints can be added to AP in order to eliminate the subtours?

⇦ ⇨

▌Introduce new variables u_i, i=1, 2, …n

▌For each pair of cities (i,j), $i \neq j$, add the constraint

$$u_i - u_j + nX_{ij} \leq n - 1$$

▌These constraints will eliminate subtours.

▌For example: $X_{12} = X_{24} = X_{41} = 1 \Rightarrow$

$u_1 - u_2 + 5 \leq 4$
$u_2 - u_4 + 5 \leq 4$
$u_4 - u_1 + 5 \leq 4$

sum: $15 \leq 12$ *infeasible!*

⇦ ⇨

Reference: Miller, Tucker, & Zemlin, J.ACM, Vol. 7 (1960), pp.326ff.

These new constraints eliminate subtours of fewer than n cities, but NOT tours of n cities:

▌Let u_i = sequence # in which city i is visited.

▌So, if $X_{ij} = 1$, we have $u_i - u_j = -1$

and so $u_i - u_j + nX_{ij} = -1 + n$

i.e., $u_i - u_j + nX_{ij} \leq n - 1$ is satisfied!

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Dimensions of model:

n(n−1)	integer variables X_{ij}
n	continuous variables u_i
2n	assignment constraints
n(n−1)	subtour elimination constraints

n (#cities)	5	10	50	100
variables				
0–1 integer	20	90	2450	9900
continuous	5	10	50	100
constraints	30	110	2550	10100

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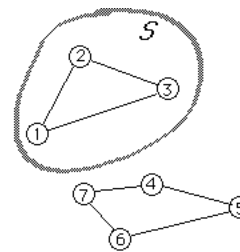**Another set of subtour elimination constraints:**

▌Let S be a nontrivial subset of the cities.

$$\sum\limits_{i \notin S} \sum\limits_{j \in S} X_{ij} \geq 1$$

insures that there is an edge in the tour which links set S to the set of cities NOT in S.

▌If we include such a constraint for every nontrivial subset, we eliminate all subtours.

▌For example, if S is the set of cities on a subtour, the constraint is violated!

⇦ ⇨

Reference: Dantzig, Fulkerson, & Johnson, O.R. Vol. 7 (1959),pp.58ff.

$$\sum\limits_{i \notin S} \sum\limits_{j \in S} X_{ij} \geq 1$$

The subtour elimination constraint is violated, since

$X_{41} = X_{51} = X_{61} = X_{71} = X_{42} = X_{52} = X_{62} = X_{72} = X_{43} = X_{53} = X_{63} = X_{73} = 0$

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Unfortunately, the number of such constraints is exponential in the number of cities ($2^n - 1$):

n (#cities)	5	10	50	100
subtour elimination constraints	31	1024	1.126×10^{15}	1.268×10^{30}

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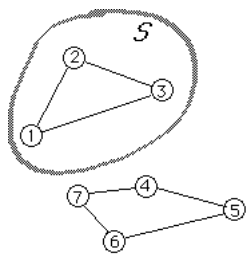
$$\sum_{i \in S} \sum_{j \in S} X_{ij} \leq |S| - 1 = 2$$

The subtour elimination constraint is violated, since there are three edges in the subtour

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Warning!

While subtours are eliminated by any of the three sets of constraints, the smaller set of constraints, i.e.,

$$u_i - u_j + nX_{ij} \leq n - 1$$ for all edges (i,j)

yields a much **weaker** lower bound in an LP relaxation!

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Let T_1 denote the set of all one-trees of the network, so that $X \in T_1$ if X is a one-tree. Then the constraint

$$X \in T_1$$

will eliminate subtours, since spanning one-trees are connected!

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Another set of subtour elimination constraints

For every nontrivial subset S,

$$\sum_{i \in S} \sum_{j \in S} X_{ij} \leq |S| - 1$$

where $|S|$ is the cardinality of the set S

The number of such constraints is again $2^n - 1$

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Summary: Subtour Elimination Constraints
(to be appended to the assignment problem)

$$\sum_{i \notin S} \sum_{j \in S} X_{ij} \geq 1$$ for all subsets S of the nodes

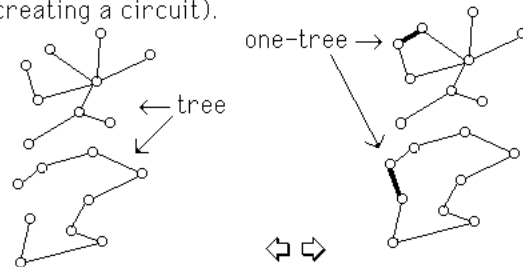$$\sum_{i \in S} \sum_{j \in S} X_{ij} \leq |S| - 1$$ for all subsets S of the nodes

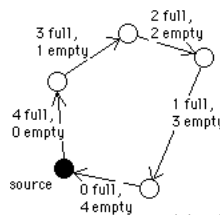$$u_i - u_j + nX_{ij} \leq n - 1$$ for all edges (i,j)

⇦ ⇨

One-Tree Constraint

A TSP tour is a special case of a "one-tree", which is a spanning tree with one additional edge included (creating a circuit).

one-tree →

← tree

⇦ ⇨

2-commodity flow model

Suppose that the salesman delivers a full container of some commodity (for example, bottled gas), and picks up an empty container, for each customer.

At all times, he will have a total of (n-1) containers (full plus empty) in his vehicle.

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Define **two** sets of **continuous** variables (one for flow of full containers, and one for flow of empty containers), plus one set of **binary** variables.

$$Y_{ij}^P = \text{flow of full containers in edge } (i,j)$$

$$Y_{ij}^Q = \text{flow of empty containers in edge } (i,j)$$

$$X_{ij} = \begin{cases} 1 & \text{if edge } (i,j) \text{ is on the route} \\ 0 & \text{otherwise} \end{cases}$$

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Constraints   (These guarantee a path from source to each node, & a return path, eliminating subtours)

$$\sum_{j=1} Y_{ij}^P - \sum_{k=1} Y_{ki}^P = \begin{cases} n-1 & \text{for } i = \text{source} \\ -1 & \text{elsewhere} \end{cases}$$

$$Y_{ij}^P \geq 0 \qquad \textit{conservation of flow for full containers}$$

$$\sum_{j=1}^n Y_{ij}^Q - \sum_{k=1}^n Y_{ki}^Q = \begin{cases} -(n-1) & \text{for } i = \text{source} \\ +1 & \text{elsewhere} \end{cases}$$

$$Y_{ij}^Q \geq 0 \qquad \textit{conservation of flow for empty containers}$$

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Since the total flow in each edge of the tour is $n-1$, we also have:

$$Y_{ij}^P + Y_{ij}^Q = (n-1) X_{ij} \quad \text{for each edge } (i,j)$$

That is, if $X_{ij} = 0$, no flow is permitted in edge (i,j)

while if $X_{ij} = 1$, the total flow is $n-1$

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Objective

$$\text{Minimize} \quad \alpha \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij} + \frac{\beta}{n-1} \sum_{i=1}^n \sum_{j=1}^n C_{ij} \left(Y_{ij}^P + Y_{ij}^Q \right)$$

where $\quad \alpha + \beta = 1, \quad \alpha \geq 0 \ \& \ \beta \geq 0$

For example $(\alpha = 1, \beta = 0)$: $\quad$ Minimize $\sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$

or $(\alpha = 0, \beta = 1)$: $\quad$ Minimize $\frac{1}{(n-1)} \sum_{i=1}^n \sum_{j=1}^n C_{ij} \left(Y_{ij}^P + Y_{ij}^Q \right)$

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This model easily incorporates

Precedence constraints

Suppose city **h** must precede city **k** on the tour. Then the number of "full containers" entering city **h** must exceed the number of "full containers" entering city **k**:

$$\sum_{i=1}^n Y_{ih}^P \geq 1 + \sum_{i=1}^n Y_{ik}^P$$

⇦ ⇨

(n−1) commodity flow model

For each city **k** (other than the source **s**), we define a commodity:

$$Y_{ij}^k = \text{flow in arc } (i,j) \text{ of commodity destined for } k$$

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Constraints

Conservation of flow for each commodity k:

$$\sum_{j=1}^n Y_{sj}^k = 1 \qquad \text{at source } s$$

$$\sum_{i=1}^n Y_{ih}^k = \sum_{j=1}^n Y_{hj}^k \qquad \text{at } h \neq s, h \neq k$$

$$\sum_{i=1}^n Y_{ik}^k = 1 \ \& \ \sum_{j=1}^n Y_{kj}^k = 0 \qquad \text{at destination } k$$

⇦ ⇨

Capacity constraints

$$0 \leq Y_{ij}^k \leq X_{ij} \qquad \text{for each arc } (i,j)$$

Assignment constraints

$$\sum_{j=1}^n X_{ij} = 1 \qquad \text{for all } i$$

$$\sum_{i=1}^n X_{ij} = 1 \qquad \text{for all } j$$

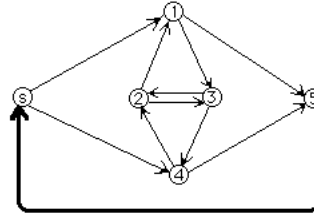$$X_{ij} \in \{0,1\} \qquad \text{for all } i \& j$$

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The feasible region of the LP relaxation of the (n-1)-commodity flow model is the <u>same</u> as that of the model with exponentially many subtour elimination constraints!

However, basic feasible solutions of this LP may be fractional, as in the following example.

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Suppose that $X_{ij} = \frac{1}{2}$ for each edge below, except for edge $(5,s)$, where $X_{5,s} = 1$.

Then X is a basic feasible solution to the LP relaxation of the assignment & subtour elimination constraints.

Can constraints be added to eliminate this solution? ⇦ ⇨

"Mutual Flow Constraints"

The following $n(n-1)/2$ constraints will eliminate the previous solution:

$$\sum_{i=1}^{n} Y_{ik}^{j} \; + \; \sum_{i=1}^{n} Y_{ij}^{k} \; = \; 1 \qquad \text{for each pair } j,k \; (j \neq k)$$

(either commodity j flows through node k, or commodity k flows through node j, but not both!)

⇦ ⇨

TSP.models

Author   Dantzig, G.B., Fulkerson, D.R., and Johnson, S.M.

Title   Solutions of a large scale traveling salesman problem

Pub.   Operations Research, Volume 2 (1954), pp. 393-410

Notes

Key   subtour elimination constraints

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TSP.models

Author   Miller, C.E., Tucker, A.W., and Zemlin, R.A.

Title   Integer programming formulation of traveling salesman problems

Pub.   J. ACM, Volume 7 (1960), pp. 326-329

Notes   n(n-1)/2 subtour elimination constraints

Key   subtour elimination constraints

⇦ ⇨

TSP.models

Author   Kusiak, Andrew and Finke, Gerd

Title   Modeling and solving the flexible forging module scheduling problem

Pub.   Engineering Optimization, Volume 12 (1987), pp. 1-12

Notes

Key   2-commodity flow model of TSP

⇦ ⇨

TSP.models

Author   Wong, Richard T.

Title   Integer programming formulations of the traveling salesman problem

Pub.   Proceedings of IEEE International Conference on Circuits & Computers, 1980

Notes

Key   multi-commodity flow model

⇦ ⇨

TSP.models

Author   Claus, A.

Title   A new formulation for the travelling salesman problem

Pub.   SIAM J. Alg. Disc. Meth., Volume 5, Number 1 (March 1984), pp. 21-25

Notes

Key   multi-commodity flow model

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